

KÄHLER C -SPACES OF CLASSICAL TYPE AND QUADRATIC BISECTIONAL CURVATURE

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ABSTRACT. In this article we give necessary and sufficient conditions for an irreducible Kähler C -space of classical type to have nonnegative or positive quadratic bisectional curvature, assuming the space is not Hermitian symmetric. The results are related to two conjectures of Li-Wu-Zheng.

Keywords: Kähler C -spaces, quadratic bisectional curvature

1. INTRODUCTION

Let (M^n, g) be a Kähler manifold of complex dimension n and let $o \in M$. M is said to have *nonnegative quadratic orthogonal bisectional curvature* at o if for any unitary frame e_i at o and real numbers ξ_i we have

$$(1.1) \quad \sum_{i,j} R_{i\bar{i}j\bar{j}}(\xi^i - \xi^j)^2 \geq 0.$$

Here $R_{i\bar{i}j\bar{j}} = R(e_i, \bar{e}_i, e_j, \bar{e}_j)$. Recall that M is said to have nonnegative bisectional curvature at o if for any $X, Y \in T_o^{(1,0)}(M)$, $R(X, \bar{X}, Y, \bar{Y}) \geq 0$, and M is said to have nonnegative *orthogonal* bisectional curvature at o if $R(X, \bar{X}, Y, \bar{Y}) \geq 0$ for all unitary pairs $X, Y \in T_o^{(1,0)}(M)$. Following [14] we abbreviate by $QB \geq 0$ for nonnegative quadratic orthogonal bisectional curvature, $B \geq 0$ for nonnegative bisectional curvature and $B^\perp \geq 0$ for nonnegative orthogonal bisectional curvature. It is obvious that $B \geq 0 \Rightarrow B^\perp \geq 0 \Rightarrow QB \geq 0$. Note that in dimension $n = 2$, the conditions $B^\perp \geq 0$ and $QB \geq 0$ are the same.

It is well-known that compact manifolds with $B \geq 0$ have been completely classified by the works [16, 17, 12, 1, 15]. By these works, we know that any compact simply connected Kähler manifold with $B \geq 0$ either biholomorphic to \mathbb{CP}^n or is isometrically biholomorphic to an irreducible compact Hermitian symmetric space of rank at least 2. While the condition $B^\perp \geq 0$ seems weaker, by the works of Chen [8] (see also [19]) and Gu-Zhang [11] we know that a compact simply connected irreducible Kähler manifold with $B^\perp \geq 0$ is also either biholomorphic to \mathbb{CP}^n or is isometrically biholomorphic to an irreducible

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compact Hermitian symmetric space of rank at least 2. Hence $B^\perp \geq 0$ does not give any new class of Kähler manifolds.

The condition $QB \geq 0$ was first considered by Wu-Yau-Zheng [21] where they proved that the boundary of the Kähler cone is non-negative under this condition. There are other interesting properties satisfied by compact Kähler manifolds with $QB \geq 0$. A fundamental property of such manifolds, implicit from earlier works [3] (see [7] for additional references) is that all harmonic real (1,1) forms are parallel. Moreover, the scalar curvature must be nonnegative and if it is irreducible then the first Chern class is positive [7].

The ultimate goal is to classify Kähler manifolds with $QB \geq 0$. For the compact case, a partial classification of the de Rham factors of the universal cover of such a manifold is given in [7]. Hence it remains to study the structure of compact simply-connected irreducible Kähler manifolds with $QB \geq 0$ and by the parallelness of real harmonic (1,1) forms mentioned above, such Kähler manifolds also have $b_2 = 1$ (see [12]). In view of the above results for $B^\perp \geq 0$, one may wonder if $QB \geq 0$ gives any new examples, even though the condition is weaker than $B^\perp \geq 0$. To address this question, recently Li, Wu and Zheng [14] were able to construct an example of simply connected irreducible compact Kähler manifold satisfying $QB \geq 0$ but not satisfying $B^\perp \geq 0$. Their example is a Kähler C -space with second Betti number $b_2 = 1$. Explicitly the example is (B_3, α_2) , see §2 for the definition. More generally, the following conjectures were raised in [14]:

Conjecture 1.1.

- (1) *Any Kähler C -space with $b_2 = 1$ satisfies $QB \geq 0$ everywhere.*
- (2) *A compact simply connected irreducible Kähler manifold (M^n, g) with $QB \geq 0$ is biholomorphic to a Kähler C -space with $b_2 = 1$.*
- (3) *In (2), if the manifold is not \mathbb{CP}^n , then g is a constant multiple of the standard metric.*

A Kähler C -space is compact simply connected Kähler manifold such that the group of holomorphic isometries acts transitively on the manifold, see [18, 13]. There is a complete classification of Kähler C -spaces with $b_2 = 1$, and this is associated with the classification of simple complex Lie algebras which are just $A_n = \mathfrak{sl}_{n+1}$, $B_n = \mathfrak{so}_{2n+1}$, $C_n = \mathfrak{sp}_{2n}$, $D_n = \mathfrak{so}_{2n}$ and the exceptional cases E_6, E_7, E_8, F_4, G_2 . Motivated by the work in [14], we prove the following results related to conjectures (1) and (3).

Theorem 1.1.

- (i) *The Kähler C -space (B_n, α_p) , $n \geq 3$, $1 < p < n$ satisfies $QB \geq 0$ if and only if $5p + 1 \leq 4n$. Moreover, $QB > 0$ if and only if $5p + 1 < 4n$.*
- (ii) *The Kähler C -space (C_n, α_p) , $n \geq 3$, $1 < p < n$ satisfies $QB \geq 0$ if and only if $5p \leq 4n + 3$. Moreover, $QB > 0$ if and only if $5p < 4n + 3$.*
- (iii) *The Kähler C -space (D_n, α_p) , $n \geq 4$, $1 < p < n - 1$ satisfies $QB \geq 0$ if and only if $5p + 3 \leq 4n$. Moreover, $QB > 0$ if and only if $5p + 3 < 4n$.*

We only consider Kähler C -spaces which are not Hermitian symmetric. According to Itoh [13], Theorem 1.1 includes all such Kähler C spaces with $b_2 = 1$ arising from the classical sequence A_n, B_n, C_n, D_n . Here $QB > 0$ means that (1.1) is a strict inequality unless all ξ_i are the same. Note that if $QB > 0$, then a small perturbation of the Kähler metric will still satisfy $QB > 0$, see Lemma 2.3 (and Remark 2.1). Hence conjecture (1) for the classical types is true only under some restrictions mentioned in Theorem 1.1, while conjecture (3) is too strong. Conjecture (2) however may still be true in general.

The organization of the paper is as follows. In §2 we will state basic properties on the condition $QB \geq 0$ and $QB > 0$, which greatly simplify the computations in later sections. In §3, §4 and §5 we will prove the main theorems for the cases B_n, C_n and D_n respectively; details for some of the calculations in these sections can be found in the appendices.

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2. BASIC FACTS

Let (M, g) be a compact Kähler C -space with transitive isometry group G , and suppose $b_2(M) = 1$. Then any real $(1, 1)$ form ρ on M is given by $\rho = c\omega + \sqrt{-1}\partial\bar{\partial}f$ for some constant c and function f where ω is the Kähler form. Now if ρ is G invariant then $\Delta_g f$ is also G invariant and hence constant on M . Thus f is constant on M and $\rho = c\omega$. In particular, g is the unique G invariant Kähler metric on M and it is Kähler Einstein. For more discussions on Kähler C -space, see [2, 13, 18, 14].

Kähler C -spaces with second Betti number $b_2 = 1$ are obtained as follows, see [4, 5, 6, 13, 14, 18]. Let G be a simply connected, simple complex Lie group, and let \mathfrak{g} be its Lie algebra let \mathfrak{h} be a Cartan subalgebra with corresponding root system Δ . Then $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$, where \mathfrak{g}_α is the root space of α . Fix an ordering on Δ with respect to a fundamental root system $\alpha_1, \dots, \alpha_l$, $l = \dim_{\mathbb{C}} \mathfrak{h}$, and let Δ^+ and Δ^- be the set of positive and negative roots respectively. Let K be the Killing form for \mathfrak{g} . We may choose root vectors $E_\alpha \in \mathfrak{g}_\alpha, \alpha \in \Delta$ such that for some constants $n_{\alpha, \beta}, \alpha, \beta \in \Delta$ we have

$$K(E_\alpha, E_{-\alpha}) = -1, \alpha \in \Delta^+; [E_\alpha, E_\beta] = n_{\alpha, \beta} E_{\alpha+\beta}$$

with $n_{\alpha, \beta} = n_{-\alpha, -\beta} \in \mathbb{R}$ and $n_{\alpha, \beta} = 0$ if $\alpha + \beta$ is not a root and $\alpha + \beta \neq 0$ (in which case $E_{\alpha+\beta}$ above is understood to be zero). Together with a suitable basis in \mathfrak{h} , they form a *Weyl canonical basis* for \mathfrak{g} . Now let $1 \leq r \leq l$ and let

$$\Delta_r^+(k) = \left\{ \sum_i n_i \alpha_i \in \Delta^+ \mid n_r = k \right\}, \quad \Delta_r^+ = \bigcup_{k>0} \Delta_r^+(k).$$

$$\mathfrak{m}_k^+ = \bigoplus_{\alpha \in \Delta_r^+(k)} \mathbb{C} E_\alpha, \quad \mathfrak{m}_k^- = \bigoplus_{\alpha \in \Delta_r^-(k)} \mathbb{C} E_\alpha, \quad \mathfrak{t} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+(0)} (\mathbb{C} E_\alpha \oplus \mathbb{C} E_{-\alpha}).$$

Let P be the subgroup whose Lie algebra is $\mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta \setminus \Delta_r^+} \mathbb{C}E_\alpha$. Then G/P is a complex homogeneous space having $b_2 = 1$. The complexified tangent space of G/P at the identity element can be identified with $\mathfrak{m}^+ \oplus \mathfrak{m}^- = (\bigoplus_{k>0} \mathfrak{m}_k^+) \oplus (\bigoplus_{k<0} \mathfrak{m}_k^+)$ so that for each $\alpha \in \Delta_r^+(k)$, E_α is a holomorphic tangent vector with conjugate $\bar{E}_\alpha = -E_{-\alpha}$. Denote the real form of G by G' . By [4, 13, 14], G' acts transitively on G/P by biholomorphisms, and the G' invariant Kähler form on G/P is given by:

Lemma 2.1.

- (i) In a Weyl canonical basis, let $\omega^\alpha, \omega^{\bar{\alpha}}$ be the dual of E_α and \bar{E}_α , $\alpha \in \Delta_r^+$. The G' invariant Kähler form on G/P is given on \mathfrak{m}^+ by

$$g = 2 \sum_{k>0} k \sum_{\alpha \in \Delta_r^+(k)} \omega^\alpha \cdot \omega^{\bar{\alpha}} = \sum_{k>0} (-kK)|_{\mathfrak{m}_k^+ \times \mathfrak{m}_k^-}.$$

- (ii) $[\mathfrak{t}, \mathfrak{m}_k^\pm] \subset \mathfrak{m}_k^\pm$, $[\mathfrak{m}_k^\pm, \mathfrak{m}_l^\pm] \subset \mathfrak{m}_{k+l}^\pm$, $[\mathfrak{m}_k^+, \mathfrak{m}_k^-] \subset \mathfrak{t}$. If $k > l > 0$, then $[\mathfrak{m}_k^+, \mathfrak{m}_l^-] \subset \mathfrak{m}_{k-l}^+$, $[\mathfrak{m}_k^-, \mathfrak{m}_l^+] \subset \mathfrak{m}_{k-l}^-$.

The Kähler C space thus obtained is denoted as (\mathfrak{g}, α_p) . Conversely, every Kähler C space with $b_2 = 1$ can be obtained by this construction. Thus the set $\{1/\sqrt{k}E_\alpha\}; \alpha \in \Delta_r^+, k \geq 1$ forms a unitary basis for the tangent space of (\mathfrak{g}, α_r) in the metric g . We call this basis as a *Weyl frame*. To compute the curvature tensor we have the following from Li-Wu-Zheng [14, Proposition 2.1].

Proposition 2.1. [Li-Wu-Zheng] On (\mathfrak{g}, α_r) , for any

$$X \in \mathfrak{m}_i^+, Y \in \mathfrak{m}_j^+, Z \in \mathfrak{m}_k^+, W \in \mathfrak{m}_l^+,$$

the components of the curvature tensor are given by

(2.1)

$$\begin{aligned} R(X, \bar{Y}, Z, \bar{W}) = & (k-j)\xi_{k-j}K([X, \bar{W}], [\bar{Y}, Z]) - \frac{kl}{i+k}K([X, Z], [\bar{Y}, \bar{W}]) \\ & + (k\xi_{i-j} + l\xi_{j-i} + l\delta_{ij}\delta_{kl})K([X, \bar{Y}], [Z, \bar{W}]), \end{aligned}$$

if $i+k = j+l$, and $R(X, \bar{Y}, Z, \bar{W}) = 0$ otherwise. Here $\xi_q = 1$ if $q > 0$ and $\xi_q = 0$ if $q \leq 0$.

Lemma 2.2. Same notations as in Proposition 2.1. Assume also that X, Y, Z, W are canonical Weyl basis vectors. Suppose $X = Y$, and $Z \neq W$, then

$$R(X, \bar{X}, Z, \bar{W}) = 0.$$

Proof. Since $i = j$, the lemma is true if $k \neq l$ by Proposition 2.1. Hence we assume $k = l$. We first assume that $k \leq i$. Then

$$R(X, \bar{X}, Z, \bar{W}) = -\frac{k^2}{i+k}K([X, Z], [\bar{X}, \bar{W}]) + kK([X, \bar{X}], [Z, \bar{W}]).$$

Let $X = E_\alpha$, $Z = E_\beta$, $W = E_\gamma$ with $\alpha, \beta, \gamma \in \Delta^+$ and $\alpha + \beta \neq 0$ and $\alpha + \gamma \neq 0$. Then $[X, Z] = n_{\alpha, \beta} E_{\alpha + \beta}$, $[X, W] = n_{\alpha, \gamma} E_{\alpha + \gamma}$. Hence

$$K([X, Z], [\bar{X}, \bar{W}]) = -n_{\alpha, \beta} n_{\alpha, \gamma} K(E_{\alpha + \beta}, E_{-\alpha - \beta}).$$

If $\alpha + \beta$ or $\alpha + \gamma$ is not a root, then $n_{\alpha, \beta} = 0$ or $n_{\alpha, \gamma} = 0$ and

$$K([X, Z], [\bar{X}, \bar{W}]) = 0.$$

Otherwise, both $E_{\alpha + \beta}$ and $E_{\alpha + \gamma}$ are canonical Weyl basis vectors and are in \mathfrak{m}_{i+k}^+ by Lemma 2.1. Since $\beta \neq \gamma$, and K is proportional to g on $\mathfrak{m}_{i+k}^+ \times \mathfrak{m}_{i+k}^-$, we also have $K([X, Z], [\bar{X}, \bar{W}]) = 0$.

Now recall that

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta^+} (\mathbb{C}E_\alpha \oplus \mathbb{C}E_{-\alpha}) \right)$$

is an orthogonal decomposition with respect to K . Now $[X, \bar{X}] \in \mathfrak{h}$. Since $\beta - \gamma \neq 0$ and $[Z, \bar{W}]$ is either zero or is a root vector of the root $\beta - \gamma$. Hence we also have $K([X, \bar{X}], [Z, \bar{W}]) = 0$. Hence the lemma is true when $k \leq i$.

Suppose $i < k$. Then it is equivalent to prove $R(X, \bar{Y}, Z, \bar{Z}) = 0$ but assuming $i > k$ and $X \neq Y$. In this case,

$$\begin{aligned} R(X, \bar{X}, Z, \bar{W}) &= R(Z, \bar{W}, X, \bar{X}) \\ &= -\frac{i^2}{i+k} K([Z, X], [\bar{W}, \bar{X}]) + iK([Z, \bar{W}], [X, \bar{X}]). \end{aligned}$$

The previous argument implies the lemma is true in this case as well. \square

Let (M^n, g) be a Kähler manifold and $p \in M$. Recall that $QB \geq 0$ at p if for any unitary frame e_A of $T_p^{(1,0)}(M)$, and any real numbers ξ_A we have

$$(2.2) \quad \sum_{A,B} R_{A\bar{A}B\bar{B}} (\xi_A - \xi_B)^2 \geq 0$$

where $R_{A\bar{A}B\bar{B}} = R(e_A, \bar{e}_A, e_B, \bar{e}_B)$. We now discuss the condition $QB \geq 0$ in more detail. We will also consider the condition $QB > 0$ at p which we define as: $QB \geq 0$ at p with strict inequality in (2.2) provided not all ξ'_A s are the same. Now define the following bilinear forms on the space $\Omega_{\mathbb{R}}^{1,1}(M)$ of real $(1,1)$ forms on M :

$$\begin{aligned} F(\eta, \sigma) &= \sum_{A,B,C,D} R_{A\bar{B}C\bar{D}} \rho^{A\bar{D}} \sigma^{C\bar{B}} = \sum_{A,B,C,D} R_{A\bar{B}C\bar{D}} \rho^{A\bar{B}} \sigma^{C\bar{D}} \\ G(\eta, \sigma) &= \frac{1}{2} (R_{A\bar{B}} g_{C\bar{D}} + R_{C\bar{D}} g_{A\bar{B}}) \rho^{A\bar{D}} \sigma^{C\bar{B}}. \end{aligned}$$

Here $\rho = \sqrt{-1} \rho_{A\bar{B}} \theta^A \wedge \bar{\theta}^B$ with $\bar{\rho}_{A\bar{B}} = \rho_{B\bar{A}}$ with local frame e_A and local coframe θ^A . $\rho^{A\bar{B}} = g^{A\bar{F}} g^{E\bar{B}} \rho_{E\bar{F}}$. $\sigma^{A\bar{B}}$ is defined for σ similarly. $R_{A\bar{B}C\bar{D}} = R(e_A, \bar{e}_B, e_C, \bar{e}_D)$.

Clearly, G and F are well defined real symmetric bilinear forms on $\Omega_{\mathbb{R}}^{1,1}(p)$ for any p . Now let θ^A be a unitary frame at any p with co-frame η_A and let

a_A be real numbers. Take $X = \sum_A \sqrt{-1} a_A \eta^A \wedge \overline{\eta^A} \Omega_{\mathbb{R}}^{1,1}(p)$. Then a simple calculation gives

$$(2.3) \quad G(X, X) - F(X, X) = \sum_A R_{A\bar{A}} a_A^2 - \sum_{A,B} R_{A\bar{A}B\bar{B}} a_A a_B = \frac{1}{2} \sum_{A,B} R_{A\bar{A}B\bar{B}} (a_A - a_B)^2.$$

The following fact was observed by Shing-Tung Yau, and communicated to us by Zheng [23].

Lemma 2.3. *At any point p we have*

- (a) $QB \geq 0$ if and only if $G - F \geq 0$.
- (b) $QB > 0$ if and only if $G - F > 0$ on $\Omega_{\mathbb{R}}^{1,1}(p) \setminus \mathbb{R}\omega(p)$.

Here $\Omega_{\mathbb{R}}^{1,1}(p) \setminus \mathbb{R}\omega(p)$ is the real $(1,1)$ forms at p which are not a multiple of the Kähler form.

Proof. We first prove (a). The fact that $G - F \geq 0$ implies $QB \geq 0$ follows immediately from (2.3) and the fact that η^A and a_A are arbitrary. Conversely, suppose $QB \geq 0$ and let X be any real $(1,1)$ form at p . Then we can always diagonalize X . Namely, there exists a unitary frame e_A with co-frame η^A such that $X = \sum_A \sqrt{-1} a_A \eta^A \wedge \overline{\eta^A}$. Now (2.3) and the assumption $QB \geq 0$ immediately implies $G(X, X) - F(X, X) \geq 0$.

Now we prove (b). The proof is basically the same in part (a) once we observe that $X \in \mathbb{R}\omega(p)$ if and only if: for every unitary frame e_A at p with co-frame η^A we have $X = c \sum_A \sqrt{-1} \eta^A \wedge \overline{\eta^A}$ for some real constant c . The fact that $G - F > 0$ on $\Omega_{\mathbb{R}}^{1,1}(p) \setminus \mathbb{R}\omega(p)$ implies $QB > 0$ now follows immediately from (2.3) and the fact that η^A and a_A are arbitrary. Conversely, suppose $QB > 0$ and let $X \in \Omega_{\mathbb{R}}^{1,1}(p) \setminus \mathbb{R}\omega(p)$. Then there exists a unitary frame e_A with co-frame η^A such that $X = \sum_A \sqrt{-1} a'_A \eta^A \wedge \overline{\eta^A}$ with a'_A s not all the same. Now (2.3) and the assumption $QB > 0$ immediately implies $G(X, X) - F(X, X) > 0$.

This concludes the proof of the Lemma. \square

Remark 2.1. Thus $QB > 0$ if and only if $G - F$ is positive in the orthogonal complement of $\mathbb{R}\omega$. In particular, if (M, g) is a compact Kähler manifold with $QB > 0$ then a Kähler metric which is a small perturbation of g will also satisfy $QB > 0$.

Remark 2.2. Viewed as an endomorphism on $\Omega_{\mathbb{R}}^{1,1}(M)$, $G - F$ is in fact the curvature term in the Weitzenböck identity for real $(1,1)$ forms: $\Delta_g - \Delta$ is given by $G - F$ up to a positive constant multiple where Δ_g is the Bochner Laplacian with respect to g and Δ is the Laplace-Beltrami operator. The standard Bochner technique and (2.3) then gives: *all real harmonic $(1,1)$ forms on M are parallel provided $QB \geq 0$, moreover, $\dim(H_{\mathbb{R}}^{1,1}(M)) = 1$ provided $QB > 0$ where $H_{\mathbb{R}}^{1,1}(M)$ is the space of real harmonic $(1,1)$ forms on M . See §1 for a reference to these facts and their implicit appearance in earlier works.*

Thus to check whether $QB \geq 0$ it is sufficient to consider $G - F \geq 0$ in a unitary frame of our choice. In the case of Kähler C -spaces, the natural choice is a Weyl frame. By Lemmas 2.2 and 2.3, we have:

Corollary 2.1. *On a Kähler C space, let $\text{Ric} = \mu g$ and let e_A be a Weyl frame. Then $QB \geq 0$ if and only if the largest eigenvalues of the quadratic forms $\sum_{A,B} R_{A\bar{A}B\bar{B}} x_A x_B$, with x_A 's real, and $\sum_{\substack{A,B,C,D; \\ A \neq B, C \neq D}} R_{A\bar{B}C\bar{D}} x_{AB} x_{CD}$, with $\overline{x_{AB}} = x_{BA}$, are at most μ .*

We will use the following simple fact to estimate the largest eigenvalue of a quadratic form. Suppose λ is an eigenvalue of an $n \times n$ matrix (a_{ij}) and let (x_1, \dots, x_n) be an eigenvector. Suppose $|x_k| = \max\{|x_i| \mid 1 \leq i \leq n\}$. Then we have

Lemma 2.4.

$$|\lambda| \leq \sum_{j=1}^n |a_{kj}|.$$

Proof. This is because

$$|\lambda x_k| = \left| \sum_{j=1}^n a_{kj} x_j \right| \leq |x_k| \sum_{j=1}^n |a_{kj}|.$$

□

3. KÄHLER C -SPACE OF TYPE (B_n, α_p)

According to [13], the Kähler C -spaces with $b_2 = 1$ of classical type which are not Hermitian symmetric spaces are (B_n, α_p) , with $n \geq 3$, $1 < p < n$, (C_n, α_p) , with $n \geq 3$, $1 < p < n$, and (D_n, α_p) , with $n \geq 4$, $1 < p < n - 1$.

In this section, we will consider the space (B_n, α_p) , with $n \geq 3$, $1 < p < n$. Here $B_n = \mathfrak{so}_{2n+1}\mathbb{C}$, consisting of complex $(2n+1) \times (2n+1)$ matrices of the form

$$\begin{bmatrix} A & B & E \\ C & D & F \\ G & H & J \end{bmatrix}, A^t = -D, B^t = -B, C^t = -C, E = -H^t, F = -G^t, J = 0,$$

where A, B have dimension $n \times n$, see [9]. Let E_{pq} be the $(2n+1) \times (2n+1)$ matrix having 1 as its (p, q) -entry and zeros for its remaining entries. A Cartan subalgebra \mathfrak{h} is generated by $H_i = E_{i,i} - E_{n+i,n+i}$, $1 \leq i \leq n$ with dual basis L_i . The corresponding roots are

$$(3.1) \quad \Delta = \{\pm L_i \pm L_j \mid 1 \leq i, j \leq n, i \neq j\} \cup \{\pm L_i \mid 1 \leq i \leq n\}$$

with eigenvectors

$$\begin{aligned}
 (3.2) \quad & L_i - L_j : X_{i,j} = E_{i,j} - E_{n+j,n+i}; i \neq j \\
 & L_i + L_j : Y_{i,j} = E_{i,n+j} - E_{j,n+i}; i \neq j \\
 & -L_i - L_j : Z_{i,j} = E_{n+i,j} - E_{n+j,i}; i \neq j \\
 & L_i : U_i = E_{i,2n+1} - E_{2n+1,n+i} \\
 & -L_i : V_i = E_{2n+1,i} - E_{n+i,2n+1}.
 \end{aligned}$$

Fundamental roots are:

$$(3.3) \quad \alpha_1 = L_1 - L_2, \alpha_2 = L_2 - L_3, \dots, \alpha_{n-1} = L_{n-1} - L_n, \alpha_n = L_n.$$

Positive roots are:

$$(3.4) \quad \Delta^+ = \{L_i + L_j\}_{i < j} \cup \{L_i - L_j\}_{i < j} \cup \{L_i\}.$$

Now

$$\begin{aligned}
 (3.5) \quad & L_i = \alpha_i + \dots + \alpha_n \\
 & L_i + L_j = \alpha_i + \dots + \alpha_{j-1} + 2\alpha_j + \dots + 2\alpha_n, \quad i < j \\
 & L_i - L_j = \alpha_i + \dots + \alpha_{j-1}, \quad i < j.
 \end{aligned}$$

Let $1 < p < n$. Recall that

$$\Delta_p^+(k) = \{\alpha \in \Delta^+ \mid \alpha = k\alpha_p + \sum_{i \neq p} m_i \alpha_i, m_i \geq 0, m_i \in \mathbb{Z}\}.$$

By (3.1) and (3.5), we have

$$(3.6) \quad \Delta_p^+(1) = \{L_i \mid 1 \leq i \leq p\} \cup \{L_i + L_j \mid i < p+1 \leq j\} \cup \{L_i - L_j \mid i < p+1 \leq j\},$$

and

$$(3.7) \quad \Delta_p^+(2) = \{L_i + L_j \mid i < j \leq p\}.$$

$$(3.8) \quad \Delta_p^+(k) = \emptyset,$$

for $k \geq 3$.

$$\begin{aligned}
 (3.9) \quad & \mathfrak{m}_1^+ = \left(\bigoplus_{a \leq p < i} \mathbb{C}X_{a,i} \right) \oplus \left(\bigoplus_{a \leq p < i} \mathbb{C}Y_{a,i} \right) \oplus \left(\bigoplus_{a \leq p} \mathbb{C}U_a \right) \\
 & \mathfrak{m}_2^+ = \bigoplus_{a < b \leq p} \mathbb{C}Y_{a,b}.
 \end{aligned}$$

We may take the Killing form to be $K(X, Y) = \frac{1}{2} \text{tr}(XY)$.

Lemma 3.1. $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}$ and $\text{tr}(E_{ij}E_{kl}) = \delta_{il}\delta_{jk}$.

Lemma 3.2.

$$\begin{aligned}
 (3.10) \quad & [X_{ij}, X_{kl}] = \delta_{jk}X_{il} - \delta_{il}X_{kj}, [X_{ij}, Y_{kl}] = \delta_{jk}Y_{il} - \delta_{jl}Y_{ik}, \\
 & [X_{ij}, Z_{kl}] = \delta_{il}Z_{jk} - \delta_{ik}Z_{jl}, [X_{ij}, U_k] = \delta_{jk}U_i, [X_{ij}, V_k] = -\delta_{ik}V_j \\
 & [Y_{ij}, Y_{kl}] = 0, [Y_{ij}, Z_{kl}] = \delta_{jk}X_{il} + \delta_{il}X_{jk} - \delta_{ik}X_{jl} - \delta_{jl}X_{ik}, \\
 & [Y_{ij}, U_k] = 0, [Y_{ij}, V_k] = -\delta_{jk}U_i + \delta_{ik}U_j, \\
 & [Z_{ij}, Z_{kl}] = 0, [Z_{ij}, U_k] = -\delta_{kj}V_i + \delta_{ik}V_j, [Z_{ij}, V_k] = 0 \\
 & [U_i, U_j] = -Y_{ij}, [U_i, V_j] = X_{ij}, [V_i, V_j] = -Z_{ij}.
 \end{aligned}$$

Note that if α, β are roots with eigenvectors E_α, E_β , then $K(E_\alpha, E_\beta) = 0$ unless $\beta = -\alpha$. Hence we have:

Lemma 3.3.

$$(3.11) \quad K(X_{ij}, X_{kl}) = \delta_{jk}\delta_{il}, K(Y_{ij}, Z_{kl}) = \delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl}, K(U_i, V_j) = \delta_{ij},$$

and

$$\begin{aligned}
 (3.12) \quad & K(X_{ij}, Y_{kl}) = K(X_{ij}, Z_{kl}) = K(X_{ij}, U_k) = K(X_{ij}, V_k) = K(Y_{ij}, Y_{kl}) \\
 & = K(Y_{ij}, U_k) = K(Y_{ij}, V_k) = K(Z_{ij}, Z_{kl}) = K(Z_{ij}, U_k) = K(Z_{ij}, V_k) \\
 & = K(U_i, U_j) = K(V_i, V_j) = 0.
 \end{aligned}$$

By the above two lemmas, $X_{ij}, Y_{ij}, Z_{ij}, U_i, V_j$ form a Weyl canonical basis mod \mathfrak{h} . Now recall $g = -K$ on $\mathfrak{m}_1^+ \times \mathfrak{m}_1^-$ and $g = -2K$ on $\mathfrak{m}_2^+ \times \mathfrak{m}_2^-$, and hence we have:

Lemma 3.4. *Consider the vectors*

- (I) : $X_{ai}, a \leq p < i; Y_{ai}, a \leq p < i; U_a, a \leq p$ (these vectors belong to \mathfrak{m}_1^+).
- (II) : $w_{ab} = \frac{1}{\sqrt{2}}Y_{ab}, a < b \leq p$ (these vectors belong to \mathfrak{m}_2^+).

The vectors in (I) and (II) form a Weyl frame. Their conjugates are $-X_{ia}, -Z_{ia}, a < p+1 \leq i; -V_a, a \leq p, -\frac{1}{\sqrt{2}}Y_{ab}, a < b \leq p$ respectively. The dimension of (B_n, α_p) is $\frac{1}{2}p(4n-3p+1)$.

From Proposition 2.1 any Lemma 2.2 we have the following lemma. The computations are routine, and details can be found in the appendices.

Lemma 3.5.

- (1) $R(X, \bar{Y}, Z, \bar{W}) = 0$, if (i) three of them belong to one group and the remaining vector is in the other group; (ii) X, Z are in one group and Y, W are in the other group; or (iii) $X = Y$, and $Z \neq W$.

The values of the remaining components of the curvature tensors are given by the following formulas together with their permutations from the properties of R . For $a, b, c, d, e, f, g, h \leq p < i, j, k, l$ we have

(2)

$$\begin{aligned}
R(X_{ai}, \bar{X}_{bj}, X_{ck}, \bar{X}_{dl}) &= R(Y_{ai}, \bar{Y}_{bj}, Y_{ck}, \bar{Y}_{dl}) = \delta_{ij}\delta_{kl}\delta_{bc}\delta_{ad} + \delta_{ab}\delta_{cd}\delta_{il}\delta_{kj}. \\
R(X_{ai}, \bar{X}_{bj}, X_{ck}, \bar{Y}_{dl}) &= R(X_{ai}, \bar{Y}_{bj}, X_{ck}, \bar{Y}_{dl}) = R(X_{ai}, \bar{Y}_{bj}, Y_{ck}, \bar{Y}_{dl}) = 0 \\
(3.13) \quad R(X_{ai}, \bar{X}_{bj}, Y_{ck}, \bar{Y}_{dl}) &= R(Y_{ai}, \bar{Y}_{bj}, X_{ck}, \bar{X}_{dl}) \\
&= -\frac{1}{2}\delta_{ik}\delta_{jl}(\delta_{bc}\delta_{ad} + \delta_{ab}\delta_{cd}) + \delta_{ij}\delta_{kl}\delta_{bc}\delta_{ad}.
\end{aligned}$$

(3)

$$(3.14) \quad R(U_a, \bar{U}_b, U_c, \bar{U}_d) = \frac{1}{2}(\delta_{bc}\delta_{ad} + \delta_{ab}\delta_{cd}).$$

$$\begin{aligned}
(3.15) \quad R(X_{ai}, \bar{U}_b, Y_{cj}, \bar{U}_d) &= R(Y_{ai}, \bar{U}_b, X_{cj}, \bar{U}_d) \\
&= -\frac{1}{2}\delta_{ij}(\delta_{bc}\delta_{ad} + \delta_{ab}\delta_{cd}).
\end{aligned}$$

$$(3.16) \quad R(X_{ai}, \bar{X}_{bj}, U_c, \bar{U}_d) = R(Y_{ai}, \bar{Y}_{bj}, U_c, \bar{U}_d) = \delta_{ij}\delta_{bc}\delta_{ad}.$$

(3.17)

$$\begin{aligned}
R(X_{ai}, \bar{U}_b, U_c, \bar{U}_d) &= R(Y_{ai}, \bar{U}_b, U_c, \bar{U}_d) = R(X_{ai}, \bar{Y}_{bj}, U_c, \bar{U}_d) = \\
&= R(X_{ai}, \bar{U}_b, X_{cj}, \bar{U}_d) = R(Y_{ai}, \bar{U}_b, Y_{cj}, \bar{U}_d) = \\
&= R(X_{ai}, \bar{X}_{bj}, X_{ck}, \bar{U}_d) = R(X_{ai}, \bar{X}_{bj}, Y_{ck}, \bar{U}_d) = R(X_{ai}, \bar{Y}_{bj}, X_{ck}, \bar{U}_d) = \\
&= R(X_{ai}, \bar{Y}_{bj}, Y_{ck}, \bar{U}_d) = R(Y_{ai}, \bar{Y}_{bj}, Y_{ck}, \bar{U}_d) = 0
\end{aligned}$$

(4)

(3.18)

$$\begin{aligned}
R(w_{ab}, \bar{w}_{cd}, w_{ef}, \bar{w}_{gh}) &= \frac{1}{2}\delta_{bd}(\delta_{fh}\delta_{ce}\delta_{ag} + \delta_{eg}\delta_{cf}\delta_{ah} - \delta_{eh}\delta_{cf}\delta_{ag} - \delta_{fg}\delta_{ce}\delta_{ah}) \\
&\quad + \frac{1}{2}\delta_{ac}(\delta_{fh}\delta_{de}\delta_{bg} + \delta_{eg}\delta_{df}\delta_{bh} - \delta_{eh}\delta_{df}\delta_{bg} - \delta_{fg}\delta_{de}\delta_{bh}) \\
&\quad + \frac{1}{2}\delta_{ad}(-\delta_{fh}\delta_{be}\delta_{cg} - \delta_{eg}\delta_{bf}\delta_{ch} + \delta_{eh}\delta_{bf}\delta_{cg} + \delta_{fg}\delta_{be}\delta_{ch}) \\
&\quad + \frac{1}{2}\delta_{bc}(-\delta_{fh}\delta_{de}\delta_{ag} - \delta_{eg}\delta_{df}\delta_{ah} + \delta_{eh}\delta_{df}\delta_{ag} + \delta_{fg}\delta_{de}\delta_{ah}).
\end{aligned}$$

(3.19)

$$\begin{aligned}
R(w_{ab}, \bar{w}_{cd}, X_{ei}, \bar{X}_{fj}) &= R(w_{ab}, \bar{w}_{cd}, Y_{ei}, Y_{fj}) \\
&= \frac{1}{2}\delta_{ij}(\delta_{bd}\delta_{ce}\delta_{af} + \delta_{ac}\delta_{de}\delta_{bf} - \delta_{ad}\delta_{ce}\delta_{bf} - \delta_{bc}\delta_{de}\delta_{af}) \\
R(w_{ab}, \bar{w}_{cd}, X_{ei}, Y_{fj}) &= 0 \\
R(w_{ab}, \bar{w}_{cd}, U_e, \bar{U}_f) &= \frac{1}{2}(\delta_{bd}\delta_{ce}\delta_{af} + \delta_{ac}\delta_{de}\delta_{bf} - \delta_{ad}\delta_{ce}\delta_{bf} - \delta_{bc}\delta_{de}\delta_{af}) \\
R(w_{ab}, \bar{w}_{cd}, X_{ei}, \bar{U}_f) &= R(w_{ab}, \bar{w}_{cd}, Y_{ei}, \bar{U}_f) = 0.
\end{aligned}$$

Corollary 3.1. *For $a, b, c, d, e, f \leq p < i, k$ we have*

$$\begin{aligned}
 R(X_{ai}, \bar{X}_{ai}, X_{ck}, \bar{X}_{ck}) &= R(Y_{ai}, \bar{Y}_{ai}, Y_{ck}, \bar{Y}_{ck}) = \delta_{ac} + \delta_{ik} \\
 R(X_{ai}, \bar{X}_{ai}, Y_{ck}, \bar{Y}_{ck}) &= R(Y_{ai}, \bar{Y}_{ai}, X_{ck}, \bar{X}_{ck}) = -\frac{1}{2}\delta_{ik}(\delta_{ac} + 1) + \delta_{ac} \\
 R(U_a, \bar{U}_a, U_c, \bar{U}_c) &= \frac{1}{2}(\delta_{ac} + 1) \\
 (3.20) \quad R(X_{ai}, \bar{X}_{ai}, U_c, \bar{U}_c) &= R(Y_{ai}, \bar{Y}_{ai}, U_c, \bar{U}_c) = \delta_{ac}. \\
 R(w_{ab}, \bar{w}_{ab}, w_{ef}, \bar{w}_{ef}) &= \frac{1}{2}(\delta_{ae} + \delta_{af} + \delta_{be} + \delta_{bf}) \\
 R(w_{ab}, \bar{w}_{ab}, X_{ei}, \bar{X}_{ei}) &= R(w_{ab}, \bar{w}_{ab}, Y_{ei}, \bar{Y}_{ei}) = \frac{1}{2}(\delta_{ae} + \delta_{be}) \\
 R(w_{ab}, \bar{w}_{ab}, U_e, \bar{U}_e) &= \frac{1}{2}(\delta_{ae} + \delta_{be}).
 \end{aligned}$$

Moreover, $\text{Ric} = (2n - p)g$.

Remark 3.1. From the corollary, if $a \neq c$ and $i = k$, then $R(X_{ai}, \bar{X}_{ai}, Y_{ck}, \bar{Y}_{ck}) = -\frac{1}{2}$. Since $p \geq 2$, there exist $a \neq c \leq p$. Hence (B_n, α_p) does not satisfy $B^\perp \geq 0$ as expected.

Next let us show that :

Lemma 3.6. *Let λ be the largest eigenvalue of the quadratic form*

$$\sum_{A,B} R_{A\bar{A}B\bar{B}} x_A x_B$$

in the Weyl frame, where x_A are reals.

- (a) $\lambda \leq 2n - p$ if and only if $5p + 1 \leq 4n$.
- (b) If $5p + 1 < 4n$, then $\lambda = (2n - p)$ iff the corresponding eigenvector satisfies $x_A = x_B$ for all A, B .
- (c) If $5p + 1 = 4n$, then there is an eigenvector with eigenvalue $(2n - p)$ such that $x_A \neq x_B$ for some $A \neq B$.

Proof. We begin with the proof of (a). Let $v : x_{ai}, y_{ai}, a \leq p < i; u_a, a \leq p; t_{ab}, a < b \leq p$ be an eigenvector corresponding to the largest eigenvalue λ for the quadratic form. Let us denote $R(X_{ai}, \bar{X}_{ai}, X_{bj}, \bar{X}_{bj})$ by $R(X_{ai}, X_{bj})$ etc. Then $P(v) = \sum_{A,B} R_{A\bar{A}B\bar{B}} x_A x_B$ is equal to:

$$\begin{aligned}
P(v) = & \sum_{a,b \leq p < i,j} R(X_{ai}, X_{bj})x_{ai}x_{bj} + \sum_{a,b \leq p < i,j} R(Y_{ai}, Y_{bj})y_{ai}y_{bj} \\
& + \sum_{a,b \leq p < i,j} R(X_{ai}, Y_{bj})x_{ai}y_{bj} + \sum_{a,b \leq p < i,j} R(Y_{ai}, X_{bj})y_{ai}x_{bj} \\
& + 2 \sum_{a,c \leq p < i} R(X_{ai}, U_c)x_{ai}u_c + 2 \sum_{a,c \leq p < i} R(Y_{ai}, U_c)y_{ai}u_c \\
(3.21) \quad & + 2 \sum_{a < b, c \leq p < i} R(w_{ab}, X_{ci})t_{ab}x_{ci} + 2 \sum_{a < b, c \leq p < i} R(w_{ab}, Y_{ci})t_{ab}y_{ci} \\
& + \sum_{a,b \leq p} R(U_a, U_b)u_a u_b + 2 \sum_{a < b, c \leq p} R(w_{ab}, U_c)t_{ab}u_c \\
& + \sum_{a < b \leq p, c < d \leq p} R(w_{ab}, w_{cd})t_{ab}t_{cd}.
\end{aligned}$$

From Corollary 3.1, we see that if we interchange x_{ai} and y_{ai} , for all a, i and obtain a vector w , then $P(v) = P(w)$ and $|v| = |w|$. We may then assume that either $x_{ai} = y_{ai}$ for all a, i , or by considering $v - w$, that $x_{ai} = -y_{ai}$ and $u_a = t_{ab} = 0$ for all a, b .

Case 1 ($x_{ai} = y_{ai}$ for all a, i): Consider the equation satisfied by v . Consider the component of v with the largest modulus which we assume to be equal to 1. Suppose the component is u_a . Then we have

$$\begin{aligned}
\lambda u_a = & \sum_{b \leq p} R(U_a, U_b)u_b + \sum_{b \leq p < i} R(X_{bi}, U_a)x_{bi} + \sum_{b \leq p < i} R(Y_{bi}, U_a)y_{bi} \\
(3.22) \quad & + \sum_{c < d \leq p} R(w_{cd}, U_a)t_{cd}.
\end{aligned}$$

Notice that the coefficients are all non-negative and the sum is just $\text{Ric}(U_a, \bar{U}_a) = 2n - p$. Hence $\lambda \leq 2n - p$. Moreover, if $\lambda = 2n - p$ then by Corollary 3.1 we must in fact have

$$(3.23) \quad x_{a,i} = y_{a,i} = u_b = t_{cd} = 1$$

for all $a, b \leq p < i$ and $c < d \leq p$.

Similarly, if w_{ab} has the largest modulus, we also have $\lambda \leq 2n - p$ and if equality holds then (3.23) is true.

Suppose now x_{ai} has the largest modulus. As above, we have

$$\begin{aligned}
\lambda x_{ai} = & \sum_{b,j} R(X_{ai}, X_{bj})x_{bj} + \sum_{b,j} R(X_{ai}, Y_{bj})y_{bj} + \sum_b R(X_{ai}, U_b)u_b \\
(3.24) \quad & + \sum_{c < d} R(X_{ai}, w_{cd})w_{cd}.
\end{aligned}$$

Since $x_{bj} = y_{bj}$, this equation is the same as:

$$(3.25) \quad \begin{aligned} \lambda x_{ai} = & \sum_{b,j} \frac{1}{2} (R(X_{ai}, X_{bj}) + R(X_{ai}, Y_{bj})) x_{bj} + \sum_{b,j} \frac{1}{2} (R(X_{ai}, X_{bj}) \\ & + R(X_{ai}, Y_{bj})) y_{bj} + \sum_b R(X_{ai}, U_b) u_b + \sum_{c < d} R(X_{ai}, w_{cd}) w_{cd}. \end{aligned}$$

By Corollary 3.1, $R(X_{ai}, X_{bj}) + R(X_{ai}, Y_{bj})$ is non-negative while the other coefficients are also non-negative. Also, the sum of the coefficients is unchanged and is $2n - p$. Hence we have $\lambda \leq 2n - p$ as before, and if equality holds then (3.23) is true by Corollary 3.1.

Similarly, if y_{ai} has the largest modulus, we also have $\lambda \leq 2n - p$ and if equality holds then (3.23) is true.

Case 2 ($x_{ai} = -y_{ai}$ and $u_a = w_{ab} = 0$ for all a, b): Suppose x_{ai} has the largest modulus. Then

$$(3.26) \quad \begin{aligned} \lambda x_{ai} = & \sum_{b,j} R(X_{ai}, X_{bj}) x_{bj} + \sum_{b,j} R(X_{ai}, Y_{bj}) y_{bj} \\ = & \sum_{b,j} (R(X_{ai}, X_{bj}) - R(X_{ai}, Y_{bj})) x_{bj} \end{aligned}$$

By Corollary 3.1, $R(X_{ai}, X_{bj}) - R(X_{ai}, Y_{bj}) \geq 0$. The sum of the coefficients is:

$$(3.27) \quad \sum_{b,j} \left(\delta_{ij} + \frac{1}{2} \delta_{ij} (\delta_{ab} + 1) \right) = p + \frac{1}{2} (p + 1) = \frac{1}{2} (3p + 1).$$

Hence if $5p + 1 \leq 4n$ then $\lambda \leq 2n - p$. Moreover, if $5p + 1 < 4n$ then $\lambda < 2n - p$.

Now suppose $5p + 1 > 4n$. Let v be such that $x_{ai} = -y_{ai} = 1$, $u_a = w_{ab} = 0$ for all a, b . Then

$$P(v) = 2 \sum_{a, b \leq p < i, j} (R(X_{ai}, X_{bj}) - R(X_{ai}, Y_{bj})) = p(n - p)(3p + 1).$$

On the other hand, $|v|^2 = 2p(n - p)$. Hence $P(v) > (2n - p)|v|^2$ because $5p + 1 > 4n$. This completes the proof of (a).

To prove (b), suppose $5p + 1 < 4n$. Then $\lambda \leq 2n - p$, and as $(2n - p)$ is always an eigenvalue we have $\lambda = 2n - p$. Let v be the corresponding eigenvector with components $x_{ai}, y_{ai}, u_a, t_{cd}$. Thus $P(v) = \lambda|v|^2$. The above proof then shows if $x_{ai} = y_{ai}$ for all a, i , then (3.23) must be true, while if $x_{ai} \neq y_{ai}$ for some a, i then we must have $\lambda < 2n - p$ which is impossible by our assumption. Hence (b) is true.

To prove (c), suppose $5p + 1 = 4n$. Then $\lambda = 2n - p$ in this case too. Let v be such that $x_{ai} = -y_{ai} = 1$, $u_a = w_{ab} = 0$ for all a, b . Then the computations above give $P(v) = \lambda|v|^2$. Since $x_{ai} \neq y_{ai}$, (c) is true. \square

Lemma 3.7. *Let λ be the largest eigenvalue of the quadratic form*

$$\sum_{A,B,C,D; A \neq B, C \neq D} R_{A\bar{B}C\bar{D}} x_{AB} x_{CD}$$

in the Weyl frame, where $x_{AB} = \overline{x_{BA}}$.

- (a) $\lambda \leq 2n - p$ if and only if $5p + 1 \leq 4n$.
- (b) If $5p + 1 < 4n$, then $\lambda < 2n - p$.
- (c) If $5p + 1 = 4n$, then there is an eigenvector with eigenvalue $(2n - p)$.

Proof. We begin with the proof of (a). Let v be an eigenvector corresponding to λ and let x_{AB} ($A \neq B$) be the component of v with the largest modulus. To prove that $\lambda \leq 2n - p$ provided $5p + 1 \leq 4n$, it is sufficient to prove that $S_{AB} = \sum_{C,D; C \neq D} |R_{A\bar{B}C\bar{D}}| \leq 2n - p$ provided $5p + 1 \leq 4n$. Note that $S_{AB} = S_{BA}$.

We estimate S_{AB} case by case.

- (i) If $A = X_{ai}$, $B = X_{bj}$, $(a, i) \neq (b, j)$, then by Lemma 3.5

(3.28)

$$\begin{aligned} S_{AB} &= \sum_{(c,k) \neq (d,l)} |R(X_{ai}, \bar{X}_{bj}, X_{ck}, \bar{X}_{dl})| + \sum_{(c,k) \neq (d,l)} |R(X_{ai}, \bar{X}_{bj}, Y_{ck}, \bar{Y}_{dl})| \\ &\quad + \sum_{(e,f) \neq (c,d); e < f; c < d} |R(X_{ai}, \bar{X}_{bj}, w_{ef}, \bar{w}_{cd})| + \sum_{c \neq d} |R(X_{ai}, \bar{X}_{bj}, U_c, \bar{U}_d)| \\ &= \sum_{(c,k) \neq (d,l)} \delta_{ij} \delta_{kl} \delta_{ac} \delta_{bd} + \delta_{ab} \delta_{cd} \delta_{il} \delta_{kj} \\ &\quad + \sum_{(ck) \neq (dl)} \left| -\frac{1}{2} \delta_{ik} \delta_{jl} (\delta_{bc} \delta_{ad} + \delta_{ab} \delta_{cd}) + \delta_{ij} \delta_{kl} \delta_{bc} \delta_{ad} \right| \\ &\quad + \sum_{(e,f) \neq (c,d); e < f; c < d} \left| \frac{1}{2} \delta_{ij} (\delta_{fd} \delta_{ca} \delta_{eb} + \delta_{ec} \delta_{da} \delta_{fb} - \delta_{ed} \delta_{ca} \delta_{fb} - \delta_{fc} \delta_{da} \delta_{eb}) \right| \\ &\quad + \sum_{c \neq d} \delta_{ij} \delta_{bc} \delta_{ad} \\ &= I + II + III + IV \end{aligned}$$

Now

$$(3.29) \quad I \leq \begin{cases} 0, & \text{if } a \neq b, i \neq j; \\ n - p, & a \neq b, i = j; \\ p, & a = b, i \neq j. \end{cases}$$

$$(3.30) \quad II \leq \begin{cases} \frac{1}{2}, & \text{if } a \neq b, i \neq j; \\ n - p - \frac{1}{2}, & a \neq b, i = j; \\ \frac{1}{2}(1 + p), & a = b, i \neq j. \end{cases}$$

$$(3.31) \quad III \leq \begin{cases} 0, & \text{if } a \neq b, i \neq j; \\ \frac{1}{2}(p-2), & a \neq b, i = j; \\ 0, & a = b, i \neq j. \end{cases}$$

$$(3.32) \quad IV \leq \begin{cases} 0, & \text{if } a \neq b, i \neq j; \\ 1, & a \neq b, i = j; \\ 0, & a = b, i \neq j. \end{cases}$$

Hence

$$(3.33) \quad S_{AB} \leq \begin{cases} \frac{1}{2}, & \text{if } a \neq b, i \neq j; \\ 2n - p - \frac{1}{2}(p+3) + 1, & a \neq b, i = j; \\ \frac{3}{2}p + \frac{1}{2}, & a = b, i \neq j. \end{cases}$$

Then since $1 < p < n$, we have $S_{AB} < 2n - p$ in the first two cases in (3.33). In the third case in (3.33) we will have $S_{AB} \leq 2n - p$ if $5p + 1 \leq 4n$ and $S_{AB} < 2n - p$ if $5p + 1 < 4n$.

(ii) $A = X_{ai}, B = Y_{bj}$. Here each term of S_{AB} vanishes and thus $S_{AB} = 0$.

(iii) $A = X_{ei}, B = w_{cd}, c < d$. Here, by Lemma 3.5 and by considering the cases that $c = e$ (which implies that $d \neq e$) and $c \neq e$, we have

$$(3.34) \quad \begin{aligned} S_{AB} &= \sum_{a < b \leq p, f \leq p} |R(X_{ei}, \bar{w}_{cd}, w_{cd}, \bar{X}_{cfj})| \\ &= \sum_{a < b \leq p, f \leq p} \left| \frac{1}{2} (\delta_{bd}\delta_{ce}\delta_{af} + \delta_{ac}\delta_{de}\delta_{bf} - \delta_{ad}\delta_{ce}\delta_{bf} - \delta_{bc}\delta_{de}\delta_{af}) \right| \leq p \end{aligned}$$

and thus $S_{AB} < 2n - p - 1 < 2n - p$ since $p < n$.

(iv) $A = Y_{ai}, B = Y_{bj}, (a, i) \neq (b, j)$. This case is similar to case (i)

(v) $A = Y_{ai}, B = w_{cd}$. This case is similar to case (iii).

(vi) $A = w_{ab}, B = w_{cd}, (a, b) \neq (c, d)$.

(case 1) $a = c, b \neq d$. By the fact that $S_{AB} = S_{BA}$, we may assume $b < d$. So $b > a = c < d$. By Lemma 3.5, the only nonzero terms in the sum of S_{AB} are:

$$(3.35) \quad \sum_{e, f \leq p < i, j} |R(w_{ab}, \bar{w}_{cd}, X_{ei}, \bar{X}_{fj})| \leq \frac{1}{2}(n - p),$$

$$(3.36) \quad \sum_{e, f \leq p < i, j} |R(w_{ab}, \bar{w}_{cd}, Y_{ei}, \bar{Y}_{fj})| \leq \frac{1}{2}(n - p),$$

$$\begin{aligned}
(3.37) \quad & \sum_{e,f,g,h \leq p; e < f, g < h} |R(w_{ab}, \bar{w}_{cd}, w_{ef}, \bar{w}_{gh})| \\
&= \frac{1}{2} \sum_{e,f,g,h \leq p; e < f, g < h} |(\delta_{fh}\delta_{de}\delta_{bg} + \delta_{eg}\delta_{df}\delta_{bh} - \delta_{eh}\delta_{df}\delta_{bg} - \delta_{fg}\delta_{de}\delta_{bh})| \\
&\leq \frac{1}{2} \left(\sum_{f,h \leq p; d < f, b < h} \delta_{fh} + \sum_{e,g \leq p; e < d, g < b} \delta_{eg} + \sum_{e,h \leq p; e < d, b < h} \delta_{eh} + \sum_{f,g \leq p; d < f, g < b} \delta_{fg} \right) \\
&= \frac{1}{2} [(p-d) + (b-1) + (d-b-1)] \\
&= \frac{1}{2} (p-2),
\end{aligned}$$

and

$$(3.38) \quad \sum_{e,f \leq p; e \neq f} |R(w_{ab}, \bar{w}_{cd}, U_e, \bar{U}_f)| = \frac{1}{2}.$$

So $S_{AB} \leq (n-p) + \frac{1}{2}(p-2) + \frac{1}{2} < 2n-p-1 < 2n-p$.

(case 2) $a \neq c, b = d$. As above, we may assume $a < c$. So $a < b = d > c$. The only nonzero terms in the sum of S_{AB} are:

$$(3.39) \quad \sum_{e,f \leq p < i,j} |R(w_{ab}, \bar{w}_{cd}, X_{ei}, \bar{X}_{fj})| \leq \frac{1}{2}(n-p),$$

$$(3.40) \quad \sum_{e,f \leq p < i,j} |R(w_{ab}, \bar{w}_{cd}, Y_{ei}, \bar{Y}_{fj})| \leq \frac{1}{2}(n-p),$$

$$\begin{aligned}
(3.41) \quad & \sum_{e,f,g,h \leq p; e < f, g < h} |R(y_{ab}, \bar{y}_{cd}, y_{ef}, \bar{y}_{gh})| \\
&= \frac{1}{2} \sum_{e,f,g,h \leq p; e < f, g < h} |(\delta_{fh}\delta_{ce}\delta_{ag} + \delta_{eg}\delta_{cf}\delta_{ah} - \delta_{eh}\delta_{cf}\delta_{ag} - \delta_{fg}\delta_{ce}\delta_{ah})| \\
&\leq \frac{1}{2} \left(\sum_{f,h \leq p; c < f, a < h} \delta_{fh} + \sum_{e,g \leq p; e < c, g < a} \delta_{eg} + \sum_{e,h \leq p; e < c, a < h} \delta_{eh} + \sum_{f,g \leq p; c < f, g < a} \delta_{fg} \right) \\
&= \frac{1}{2} [(p-c) + (a-1) + (c-a-1)] \\
&= \frac{1}{2} (p-2),
\end{aligned}$$

and

$$(3.42) \quad \sum_{e,f \leq p; e \neq f} |R(w_{ab}, \bar{w}_{cd}, U_e, \bar{U}_f)| = \frac{1}{2}.$$

So in this case $S_{AB} = (n - p) + \frac{1}{2}(p - 2) + \frac{1}{2} < 2n - p$ because $p < n$.

(case 3) $a \neq c, b \neq d$. As above, we may assume $b < d$. The only nonzero terms in the sum of S_{AB} are:

$$(3.43) \quad \sum_{e,f \leq p < i,j} |R(w_{ab}, \bar{w}_{cd}, X_{ei}, \bar{X}_{fj})| \leq \frac{1}{2}(n - p),$$

Similarly

$$(3.44) \quad \sum_{e,f \leq p < i,j} |R(w_{ab}, \bar{w}_{cd}, Y_{ei}, \bar{Y}_{fj})| \leq \frac{1}{2}(n - p),$$

$$(3.45) \quad \begin{aligned} & \sum_{e,f,g,h \leq p < e < f, g < h} |R(w_{ab}, \bar{w}_{cd}, w_{ef}, \bar{w}_{gh})| \\ &= \frac{1}{2} \sum_{e,f,g,h \leq p < e < f, g < h} |(-\delta_{fh}\delta_{de}\delta_{ag} - \delta_{eg}\delta_{df}\delta_{ah} + \delta_{eh}\delta_{df}\delta_{ag} + \delta_{fg}\delta_{de}\delta_{ah})| \\ &\leq \frac{1}{2} \left(\sum_{f,h \leq p; d < f, a < h} \delta_{fh} + \sum_{e,g \leq p; e < d, g < a} \delta_{eg} + \sum_{e,h \leq p; e < d, a < h} \delta_{eh} + \sum_{f,g \leq p; d < f, g < a} \delta_{fg} \right) \\ &= \frac{1}{2} [(p - d) + (a - 1) + (d - a - 1)] \\ &= \frac{1}{2}(p - 2), \end{aligned}$$

and

$$(3.46) \quad \sum_{e,f \leq p; e \neq f} |R(w_{ab}, \bar{w}_{cd}, U_e, \bar{U}_f)| = \frac{1}{2}.$$

So we also have $S_{AB} < 2n - p$.

(vii) $A = U_a, B = U_b, a \neq b$. We may assume that $a < b$. The only nonzero terms in the sum of S_{AB} are:

$$(3.47) \quad \sum_{c,d \leq p; c \neq d} |R(U_a, \bar{U}_b, U_c, \bar{U}_d)| = \frac{1}{2},$$

$$(3.48) \quad \sum_{c,d \leq p < i,j; (c,i) \neq (d,j)} |R(U_a, \bar{U}_b, X_{ci}, \bar{X}_{dj})| = n - p,$$

$$(3.49) \quad \sum_{c,d \leq p < i,j; (c,i) \neq (d,j)} |R(U_a, \bar{U}_b, Y_{ci}, \bar{Y}_{dj})| = n - p,$$

and

$$(3.50) \quad \sum_{c,d,e,f \leq p; (c,d) \neq (e,f)} |R(U_a, \bar{U}_b, w_{cd}, \bar{w}_{ef})| \begin{cases} \leq \frac{1}{2}(p-2), & \text{if } p > 2; \\ = 0, & \text{if } p = 2. \end{cases}$$

because $a < b$. Hence $S_{AB} \leq 2(n-p) + \frac{1}{2}(p-1) < 2n-p$, if $p > 2$, and $S_{AB} = 2(n-p) + \frac{1}{2} < 2n-p$ if $p = 2$.

(viii) $A = X_{ai}$, $B = U_b$. The only nonzero terms in the sum of S_{AB} are:

$$(3.51) \quad \sum_{c,d \leq p < j} |R(X_a, \bar{U}_b, U_d, \bar{X}_{cj})| \leq p,$$

$$(3.52) \quad \sum_{c,d \leq p < j} |R(X_a, \bar{U}_b, \bar{Y}_{cj}, \bar{U}_d)| \leq \frac{1}{2}(1+p).$$

Hence $S_{AB} \leq \frac{1}{2}(3p+1) \leq 2n-p$ if $4n \geq 5p+1$ and $S_{AB} \leq \frac{1}{2}(3p+1) < 2n-p$ if $4n > 5p+1$.

(ix) $A = Y_{ai}$, $B = U_b$. This case is similar to (vii).

(x) $A = w_{ab}$, $B = U_c$. Here $S_{AB} = 0$.

We have proved above that $\lambda \leq 2n-p$ provided $5p+1 \leq 4n$. Conversely, suppose $5p+1 > 4n$. Define the vector v by: $x_{ai,aj} = 1 = -y_{ai,aj}$ if $i \neq j$, all other x 's, y 's and all t are zeros. Notice that $x_{ai,aj}$ etc are symmetric and real. $F(v) = \sum_{A,B,C,D} R_{AB\bar{C}\bar{D}} x_{AB} x_{CD}$ is equal to

$$(3.53) \quad \begin{aligned} F(v) &= \sum_{a,c \leq p < i,j,k,l; i \neq j, k \neq l} \left[2R(X_{ai}, \bar{X}_{aj}, X_{ck}, \bar{X}_{cl}) - 2R(X_{ai}, \bar{X}_{aj}, Y_{ck}, \bar{Y}_{cl}) \right] \\ &= 2p^2(n-p)(n-p-1) + p(n-p)(n-p-1) + p^2(n-p)(n-p-1) \\ &= p(n-p)(n-p-1)(3p+1). \end{aligned}$$

On the other hand,

$$\sum_{a \leq p < i,j; i \neq j} (x_{ai,aj}^2 + y_{ai,aj}^2) = 2p(n-p)(n-p-1).$$

Hence $F(v)/|v|^2 = (3p+1)/2$, and we have $F(v) > (2n-p)|v|^2$ if $5p+1 > 4n$. This completes the proof of (a).

To prove (b), suppose $4n > 5p + 1$. Then by the computations from (a), we conclude that $\lambda < 2n - p$. To prove (c), note that in the above example we in fact have $F(v) = (2n - p)|v|^2$ iff $5p + 1 = 4n$. \square

Theorem 3.1. *The Kähler C-space (B_n, α_p) , $n \geq 3$, $1 < p < n$ satisfies $QB \geq 0$ if and only if $5p + 1 \leq 4n$. Moreover, $QB > 0$ if and only if $5p + 1 < 4n$.*

Proof. The first statement follows from part (a) of Lemmas 3.6 and 3.7 and Corollary 2.1. For the second statement, note that $QB > 0$ iff $G - F > 0$ on $\Omega_{\mathbb{R}}^{1,1}(p) \setminus \mathbb{R}\omega(p)$ by Lemma 2.3. On the other hand, here $G = (2n - p)Id$ on $\Omega_{\mathbb{R}}^{1,1}(p) \setminus \mathbb{R}\omega(p)$, and thus by parts (b) and (c) of Lemmas 3.6 and Lemma 3.7 we have $G - F > 0$ iff $5p + 1 < 4n$. Thus we have $QB > 0$ if and only if $5p + 1 < 4n$. \square

4. KÄHLER C-SPACE OF TYPE (C_n, α_p)

We will consider the space (C_n, α_p) , with $n \geq 3$, $1 < p < n$. Here $C_n = \mathfrak{sp}_{2n}\mathbb{C}$, consisting of complex $2n \times 2n$ matrices of the form

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, C^t = C, B^t = B, D = -A^t.$$

where A, B have dimension $n \times n$, see [9]. Let E_{ij} be the $(2n) \times (2n)$ matrix with (i, j) -entry equal to 1 and all other entries equal to zero. A Cartan subalgebra \mathfrak{h} is generated by $H_i = E_{ii} - E_{n+i, n+i}$, $1 \leq i \leq n$ with dual basis L_i . The corresponding roots are

$$(4.1) \quad \Delta = \{\pm L_i \pm L_j | 1 \leq i, j \leq n\}$$

with eigenvectors

$$(4.2) \quad \begin{aligned} L_i - L_j : X_{ij} &= E_{ij} - E_{n+j, n+i}; i \neq j \\ L_i + L_j : Y_{ij} &= E_{i, n+j} + E_{j, n+i}; i \neq j \\ -L_i - L_j : Z_{i,j} &= E_{n+i, j} + E_{n+j, i}; i \neq j \\ 2L_i : U_i &= E_{i, n+i} \\ -2L_i : V_i &= E_{n+i, i}. \end{aligned}$$

Fundamental roots are:

$$(4.3) \quad \alpha_1 = L_1 - L_2, \alpha_2 = L_2 - L_3, \dots, \alpha_{n-1} = L_{n-1} - L_n, \alpha_n = 2L_n.$$

Positive roots are:

$$(4.4) \quad \Delta^+ = \{L_i + L_j\}_{i < j} \cup \{L_i - L_j\}_{i < j}.$$

Now

$$(4.5) \quad \begin{aligned} L_i - L_j &= \alpha_i + \dots + \alpha_{j-1}, \quad i < j \leq n, \\ 2L_i &= 2(\alpha_i + \dots + \alpha_{j-1} + \alpha_j + \dots + \alpha_{n-1}) + \alpha_n, \quad i < n, \quad 2L_n = \alpha_n, \\ L_i + L_j &= \alpha_i + \dots + \alpha_{j-1} + 2(\alpha_j + \dots + \alpha_{n-1}) + \alpha_n, \quad i < j \leq n. \end{aligned}$$

Let $1 < p < n$. Recall that

$$\Delta_p^+(k) = \{\alpha \in \Delta^+ \mid \alpha = k\alpha_p + \sum_{i \neq p} m_i \alpha_i, m_i \geq 0, m_i \in \mathbb{Z}\}.$$

By (4.1) and (4.5) we have

$$(4.6) \quad \Delta_p^+(1) = \{L_i \pm L_j \mid i \leq p < j\},$$

and

$$(4.7) \quad \Delta_p^+(2) = \{2L_i \mid i \leq p\} \cup \{L_i + L_j \mid i < j \leq p\}$$

$$(4.8) \quad \Delta_p^+(k) = \emptyset,$$

for $k \geq 3$.

$$(4.9) \quad \begin{aligned} \mathfrak{m}_1^+ &= \left(\bigoplus_{a \leq p < j} \mathbb{C}X_{aj} \right) \oplus \left(\bigoplus_{a \leq p < j} \mathbb{C}Y_{aj} \right) \\ \mathfrak{m}_2^+ &= \left(\bigoplus_{a \leq p} \mathbb{C}U_a \right) \oplus \left(\bigoplus_{a < b \leq p} \mathbb{C}Y_{ab} \right) \end{aligned}$$

We may take the Killing form to be $K(X, Y) = \frac{1}{2} \text{tr}(XY)$.

Lemma 4.1. $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}$ and $\text{tr}(E_{ij}E_{kl}) = \delta_{il}\delta_{jk}$.

Lemma 4.2.

$$(4.10) \quad \begin{aligned} [X_{ij}, X_{kl}] &= \delta_{jk}X_{il} - \delta_{il}X_{kj}, [X_{ij}, Y_{kl}] = \delta_{jk}Y_{il} + \delta_{jl}Y_{ik}, \\ [X_{ij}, Z_{kl}] &= -\delta_{il}Z_{jk} - \delta_{ik}Z_{jl}, [X_{ij}, U_k] = \delta_{jk}Y_{ik}, [X_{ij}, V_k] = -\delta_{ik}Z_{jk} \\ [Y_{ij}, Y_{kl}] &= 0, [Y_{ij}, Z_{kl}] = \delta_{jk}X_{il} + \delta_{il}X_{jk} + \delta_{ik}X_{jl} + \delta_{jl}X_{ik}, \\ [Y_{ij}, U_k] &= 0, [Y_{ij}, V_k] = \delta_{jk}X_{ik} + \delta_{ik}X_{jk}, \\ [Z_{ij}, Z_{kl}] &= 0, [Z_{ij}, U_k] = -\delta_{jk}X_{ki} - \delta_{ik}X_{kj}, [Z_{ij}, V_k] = 0 \\ [U_i, U_j] &= [V_i, V_j] = 0, [U_i, V_j] = \delta_{ij}X_{ij}. \end{aligned}$$

Note that if α, β are roots with eigenvectors E_α, E_β , then $K(E_\alpha, E_\beta) = 0$ unless $\beta = -\alpha$. Hence we have:

Lemma 4.3.

$$(4.11) \quad K(X_{ij}, X_{kl}) = \delta_{jk}\delta_{il}, K(Y_{ij}, Z_{kl}) = \delta_{jk}\delta_{il} + \delta_{ik}\delta_{jl}, K(U_i, V_k) = \frac{1}{2}\delta_{ik}$$

and

$$(4.12) \quad \begin{aligned} K(X_{ij}, Y_{kl}) &= K(X_{ij}, Z_{kl}) = K(X_{ij}, U_k) = K(X_{ij}, V_k) = K(Y_{ij}, Y_{kl}) \\ &= K(Y_{ij}, U_k) = K(Y_{ij}, V_k) = K(Z_{ij}, Z_{kl}) = K(Z_{ij}, U_k) = K(Z_{ij}, V_k) \\ &= K(U_i, U_k) = K(V_i, V_k) = 0. \end{aligned}$$

By the above two lemmas, $X_{ij}, Y_{ij}, Z_{ij}, \frac{1}{\sqrt{2}}U_i, \frac{1}{\sqrt{2}}V_j$ form a Weyl canonical basis mod \mathfrak{h} . Now recall $g = -K$ on $\mathfrak{m}_1^+ \times \mathfrak{m}_1^-$ and $g = -2K$ on $\mathfrak{m}_2^+ \times \mathfrak{m}_2^-$, and hence

Lemma 4.4. *Consider the vectors*

- (I) : $X_{ai}, Y_{ai}; a \leq p < i \leq n$ (these vectors belong to \mathfrak{m}_1^+).
- (II) : $U_a, \frac{1}{\sqrt{2}}Y_{bc} = w_{bc}; b < c \leq p; a \leq p$ (these vectors belong to \mathfrak{m}_2^+).

The vectors in (I) and (II) form a Weyl frame. Their conjugates are $-X_{ia}, -Z_{ia}; -V_a, -\frac{1}{\sqrt{2}}Z_{ba}$, respectively. The dimension of (C_n, α_p) is $\frac{1}{2}p(4n - 3p + 1)$.

From Proposition 2.1 and Lemma 2.2, we have the following lemma. The computations are routine, and details can be found in the appendices.

Lemma 4.5.

- (1) $R(X, \bar{Y}, Z, \bar{W}) = 0$, if (i) three of them belong to one group and the remaining vector is in the other group; (ii) X, Z are in one group and Y, W are in the other group; or (iii) $X = Y$, and $Z \neq W$.

The values of the remaining components of the curvature tensors are given by the following formulas together with their permutations from the properties of R . For $a, b, c, d, e, f, g, h \leq p < i, j, k, l$ we have

(2)

$$\begin{aligned}
 R(X_{ai}, \bar{X}_{bj}, X_{ck}, \bar{X}_{dl}) &= R(Y_{ai}, \bar{Y}_{bj}, Y_{ck}, \bar{Y}_{dl}) = \delta_{ij}\delta_{kl}\delta_{bc}\delta_{ad} + \delta_{ab}\delta_{cd}\delta_{jk}\delta_{il}. \\
 R(X_{ai}, \bar{X}_{bj}, X_{ck}, \bar{Y}_{dl}) &= R(X_{ai}, \bar{Y}_{bj}, X_{ck}, \bar{Y}_{dl}) = R(X_{ai}, \bar{Y}_{bj}, Y_{ck}, \bar{Y}_{dl}) = 0 \\
 (4.13) \quad R(X_{ai}, \bar{X}_{bj}, Y_{ck}, \bar{Y}_{dl}) &= R(Y_{ai}, \bar{Y}_{bj}, X_{ck}, \bar{X}_{dl}) \\
 &= \frac{1}{2}\delta_{ik}\delta_{jl}(\delta_{bc}\delta_{ad} - \delta_{ab}\delta_{cd}) + \delta_{ij}\delta_{kl}\delta_{bc}\delta_{ad}.
 \end{aligned}$$

(3)

$$(4.14) \quad R(U_a, \bar{U}_b, U_c, \bar{U}_d) = 2\delta_{ab}\delta_{cd}\delta_{bc}\delta_{ad}.$$

$$(4.15) \quad R(U_a, \bar{U}_b, X_{ci}, \bar{X}_{dj}) = R(U_a, \bar{U}_b, Y_{ci}, \bar{Y}_{dj}) = \delta_{ab}\delta_{ij}\delta_{ad}\delta_{bc}$$

$$(4.16) \quad R(U_a, \bar{U}_b, X_{ci}, \bar{Y}_{dj}) = 0.$$

(4)

(4.17)

$$\begin{aligned}
 R(w_{ab}, \bar{w}_{cd}, w_{ef}, \bar{w}_{gh}) &= \frac{1}{2}[\delta_{bd}(\delta_{fh}\delta_{ag}\delta_{ce} + \delta_{eg}\delta_{ah}\delta_{cf} + \delta_{eh}\delta_{ag}\delta_{cf} + \delta_{fg}\delta_{ah}\delta_{ce}) \\
 &\quad + \delta_{ac}(\delta_{fh}\delta_{bg}\delta_{de} + \delta_{eg}\delta_{bh}\delta_{df} + \delta_{eh}\delta_{bg}\delta_{df} + \delta_{fg}\delta_{bh}\delta_{de}) \\
 &\quad + \delta_{ad}(\delta_{fh}\delta_{bg}\delta_{ce} + \delta_{eg}\delta_{bh}\delta_{cf} + \delta_{eh}\delta_{bg}\delta_{cf} + \delta_{fg}\delta_{bh}\delta_{ce}) \\
 &\quad + \delta_{bc}(\delta_{fh}\delta_{ag}\delta_{de} + \delta_{eg}\delta_{ah}\delta_{df} + \delta_{eh}\delta_{ag}\delta_{df} + \delta_{fg}\delta_{ah}\delta_{de})]
 \end{aligned}$$

(4.18)

$$R(U_a, \bar{w}_{bc}, w_{de}, \bar{w}_{fg}) = \frac{1}{\sqrt{2}} \delta_{ab} (\delta_{eg} \delta_{af} \delta_{cd} + \delta_{df} \delta_{ag} \delta_{ce} + \delta_{dg} \delta_{af} \delta_{ce} + \delta_{ef} \delta_{ag} \delta_{cd}) \\ + \frac{1}{\sqrt{2}} \delta_{ac} (\delta_{eg} \delta_{af} \delta_{bd} + \delta_{df} \delta_{ag} \delta_{be} + \delta_{dg} \delta_{af} \delta_{be} + \delta_{ef} \delta_{ag} \delta_{bd})$$

(4.19)

$$R(w_{ab}, \bar{w}_{cd}, X_{ei}, \bar{X}_{fj}) = R(w_{ab}, \bar{w}_{cd}, Y_{ei}, \bar{Y}_{fj}) \\ = \frac{1}{2} \delta_{ij} (\delta_{bd} \delta_{af} \delta_{ce} + \delta_{ac} \delta_{bf} \delta_{de} + \delta_{ad} \delta_{bf} \delta_{ce} + \delta_{bc} \delta_{af} \delta_{de})$$

(4.20)

$$R(w_{ab}, \bar{w}_{cd}, X_{ei}, \bar{Y}_{fj}) = 0.$$

(4.21)

$$R(U_a, \bar{U}_b, w_{cd}, \bar{w}_{ef}) = \delta_{ab} (\delta_{df} \delta_{ae} \delta_{bc} + \delta_{ce} \delta_{af} \delta_{bd})$$

(4.22)

$$R(U_a, \bar{w}_{cd}, U_b, \bar{w}_{ef}) = (\delta_{ac} \delta_{fb} \delta_{ae} \delta_{db} + \delta_{ad} \delta_{eb} \delta_{af} \delta_{cb})$$

(4.23)

$$R(U_a, \bar{U}_b, U_c, w_{de}) = 0.$$

(4.24)

$$R(U_a, \bar{w}_{bc}, X_{di}, \bar{X}_{ej}) = R(U_a, \bar{w}_{bc}, Y_{di}, \bar{Y}_{ej}) = \frac{1}{\sqrt{2}} \delta_{ij} (\delta_{ab} \delta_{ae} \delta_{cd} + \delta_{ac} \delta_{ae} \delta_{bd})$$

(4.25)

$$R(U_a, \bar{w}_{bc}, X_{di}, \bar{Y}_{ej}) = R(w_{ab}, \bar{U}_c, X_{di}, \bar{Y}_{ej}) = 0.$$

Corollary 4.1. For $a, b, c, d, e, f \leq p < i, k$ we have

$$R(X_{ai}, \bar{X}_{ai}, X_{ck}, \bar{X}_{ck}) = R(Y_{ai}, \bar{Y}_{ai}, Y_{ck}, \bar{Y}_{ck}) = \delta_{ac} + \delta_{ik} \\ R(X_{ai}, \bar{X}_{ai}, Y_{ck}, \bar{Y}_{ck}) = R(Y_{ai}, \bar{Y}_{ai}, X_{ck}, \bar{X}_{ck}) = \frac{1}{2} \delta_{ik} (\delta_{ac} - 1) + \delta_{ac} \\ R(U_a, \bar{U}_a, U_c, \bar{U}_c) = 2\delta_{ac} \\ R(X_{ai}, \bar{X}_{ai}, U_c, \bar{U}_c) = R(Y_{ai}, \bar{Y}_{ai}, U_c, \bar{U}_c) = \delta_{ac}. \\ R(w_{ab}, \bar{w}_{ab}, w_{ef}, \bar{w}_{ef}) = \frac{1}{2} (\delta_{ae} + \delta_{af} + \delta_{be} + \delta_{bf}); \quad a < b, e < f. \\ R(w_{ab}, \bar{w}_{ab}, X_{ei}, \bar{X}_{ei}) = R(w_{ab}, \bar{w}_{ab}, Y_{ei}, \bar{Y}_{ei}) = \frac{1}{2} (\delta_{ae} + \delta_{be}); \quad a < b. \\ R(w_{ab}, \bar{w}_{ab}, U_e, \bar{U}_e) = (\delta_{ae} + \delta_{be}); \quad a < b.$$

Moreover, $\text{Ric} = (2n - p + 1)g$.

Remark 4.1. As in the case of (B_n, α_p) , from the corollary we can see that (C_n, α_p) does not satisfy $B^\perp \geq 0$.

From the corollary, the same proof as for Lemma 3.6 gives

Lemma 4.6. *Let λ be the largest eigenvalue of the quadratic form*

$$\sum_{A,B} R_{A\bar{A}B\bar{B}} x_A x_B$$

in the Weyl frame, where x_A are reals.

- (a) $\lambda \leq 2n - p + 1$ if and only if $5p \leq 4n + 3$.
- (b) If $5p < 4n + 3$, then $\lambda = (2n - p)$ iff the corresponding eigenvector has $x_A = x_B$ for all A, B .
- (c) If $5p = 4n + 3$, then there is an eigenvector with eigenvalue $(2n - p)$ such that $x_A \neq x_B$ for some $A \neq B$.

Proof. Part (a): the argument is identical to the proof of Lemma 3.6 (a) except that: in Case 1 we use that for any B the coefficients in $\sum_A R(A, B)x_A$ must add to $2n - p + 1$ (instead of $2n - p$), in Case 2 we use that

$$\sum_{b,j} (R(X_{ai}, X_{bj}) - R(X_{ai}, Y_{bj})) = \sum_{b,j} \left(\frac{3}{2} \delta_{ij} - \frac{1}{2} \delta_{ij} \delta_{ab} \right) = \frac{3}{2} p - \frac{1}{2}$$

(instead of $\frac{3}{2} p + \frac{1}{2}$).

Parts (b) and (c): the argument is identical to the corresponding proofs for Lemma 3.6.

□

Lemma 4.7. *Let λ be the largest eigenvalue of the quadratic form*

$$\sum_{A,B,C,D; A \neq B, C \neq D} R_{A\bar{B}C\bar{D}} x_{AB} x_{CD}$$

in the Weyl frame, where $x_{AB} = \overline{x_{BA}}$.

- (a) $\lambda \leq 2n - p + 1$ if and only if $5p \leq 4n + 3$.
- (b) If $5p < 4n + 3$, then $\lambda < 2n - p + 1$.
- (c) If $5p = 4n + 3$, then there is an eigenvector with eigenvalue $(2n - p)$.

Proof. We begin with the proof of (a). Let v be an eigenvector corresponding to λ and let x_{AB} ($A \neq B$) be the component of v with the largest modulus. To prove that $\lambda \leq 2n - p + 1$ provided $5p \leq 4n + 3$, it is sufficient to prove that $S_{AB} = \sum_{C,D; C \neq D} |R_{A\bar{B}C\bar{D}}| \leq 2n - p + 1$ provided $5p \leq 4n + 3$. Note that $S_{AB} = S_{BA}$.

We estimate S_{AB} case by case.

(i) $A = X_{ai}$, $B = X_{bj}$, $(a, i) \neq (b, j)$. We may assume that $a \leq b$. By Lemma 4.6,

$$\begin{aligned}
(4.27) \quad S_{AB} &= \sum_{(c,k) \neq (d,l)} |R(X_{ai}, \bar{X}_{bj}, X_{ck}, \bar{X}_{dl})| + \sum_{(c,k) \neq (d,l)} |R(X_{ai}, \bar{X}_{bj}, Y_{ck}, \bar{Y}_{dl})| \\
&\quad + \sum_{c;d < e} |R(X_{ai}, \bar{X}_{bj}, U_c, \bar{w}_{de})| + |R(X_{ai}, \bar{X}_{bj}, w_{de}, \bar{U}_c)| \\
&\quad + \sum_{c \neq d} |R(X_{ai}, \bar{X}_{bj}, U_c, \bar{U}_d)| + \sum_{(c,d) \neq (e,f); c < d, e < f} |R(X_{ai}, \bar{X}_{bj}, w_{cd}, \bar{w}_{ef})| \\
&= \sum_{(c,k) \neq (d,l)} \delta_{ij} \delta_{kl} \delta_{bc} \delta_{ad} + \delta_{ab} \delta_{cd} \delta_{jk} \delta_{il} \\
&\quad + \sum_{(ck) \neq (dl)} \left| \frac{1}{2} \delta_{ik} \delta_{jl} (\delta_{bc} \delta_{ad} - \delta_{ab} \delta_{cd}) + \delta_{ij} \delta_{kl} \delta_{bc} \delta_{ad} \right| \\
&\quad + \sum_{c;d < e} \frac{1}{\sqrt{2}} \delta_{ij} [\delta_{bc} (\delta_{cd} \delta_{ae} + \delta_{ce} \delta_{da}) + \delta_{ca} (\delta_{cd} \delta_{eb} + \delta_{ce} \delta_{db})] \\
&\quad + \sum_{c \neq d} \delta_{cd} \delta_{ij} \delta_{cb} \delta_{ad} \\
&\quad + \sum_{(cd) \neq (ef)} \frac{1}{2} \delta_{ij} (\delta_{de} \delta_{bc} \delta_{af} + \delta_{cf} \delta_{bd} \delta_{ae} + \delta_{ce} \delta_{bd} \delta_{af} + \delta_{df} \delta_{bc} \delta_{ea}) \\
&= I + II + III + IV + V
\end{aligned}$$

$$\begin{aligned}
(4.28) \quad I &= \sum_{k, (b,k) \neq (a,k)} \delta_{ij} + \sum_{(c,j) \neq (d,i)} \delta_{ab} \delta_{cd} \\
&\leq \begin{cases} 0, & \text{if } a \neq b, i \neq j; \\ n - p, & a \neq b, i = j; \\ p, & a = b, i \neq j. \end{cases}
\end{aligned}$$

$$\begin{aligned}
(4.29) \quad II &= \sum_{(c,k) \neq (d,l)} \left| \frac{1}{2} \delta_{ik} \delta_{jl} (\delta_{bc} \delta_{ad} - \delta_{ab} \delta_{cd}) + \delta_{ij} \delta_{kl} \delta_{bc} \delta_{ad} \right| \\
&\leq \begin{cases} \frac{1}{2}, & \text{if } a \neq b, i \neq j; \\ n - p + \frac{1}{2}, & a \neq b, i = j; \\ \frac{1}{2}(p - 1), & a = b, i \neq j. \end{cases}
\end{aligned}$$

$$\begin{aligned}
 (4.30) \quad III &= \sum_{c;d < e} \frac{1}{\sqrt{2}} \delta_{ij} [\delta_{bc}(\delta_{cd}\delta_{ae} + \delta_{ce}\delta_{da}) + \delta_{ca}(\delta_{cd}\delta_{eb} + \delta_{ce}\delta_{db})] \\
 &\leq \begin{cases} 0, & \text{if } a \neq b, i \neq j; \\ \sqrt{2}, & a \neq b, i = j; \\ 0, & a = b, i \neq j. \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 (4.31) \quad IV &= \sum_{c \neq d} \delta_{cd} \delta_{ij} \delta_{cb} \delta_{ad} \\
 &= \begin{cases} 0, & \text{if } a \neq b, i \neq j; \\ 0, & a \neq b, i = j; \\ 0, & a = b, i \neq j. \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 (4.32) \quad V &= \sum_{(c,d) \neq (e,f); c < d, e < f} \frac{1}{2} \delta_{ij} (\delta_{de} \delta_{bc} \delta_{af} + \delta_{cf} \delta_{bd} \delta_{ae} + \delta_{ce} \delta_{bd} \delta_{af} + \delta_{df} \delta_{bc} \delta_{ea}) \\
 &\leq \begin{cases} 0, & \text{if } a \neq b, i \neq j; \\ \frac{1}{2}(p-2), & a \neq b, i = j; \text{ hence } a < b \\ 0, & a = b, i \neq j. \end{cases}
 \end{aligned}$$

Hence in this case

$$(4.33) \quad S_{AB} = \begin{cases} \frac{1}{2}, & \text{if } a \neq b, i \neq j; \\ 2n - \frac{3}{2}p - \frac{1}{2} + \sqrt{2}, & a \neq b, i = j; \\ \frac{3}{2}p - \frac{1}{2}, & a = b, i \neq j, \end{cases}$$

and in the first two cases we have $S_{AB} < 2n - p + 1$ since $1 < p < n$ while in the third case we $S_{AB} \leq 2n - p + 1$ iff $5p \leq 4n + 3$ while $S_{AB} < 2n - p + 1$ iff $5p < 4n + 3$.

(ii) $A = X_{ai}$, $B = Y_{bj}$. In this case $S_{AB} = 0$.

(iii) $A = X_{ai}$, $B = U_b$. Note that $R(X_{ai}, \overline{U_b}, Z, \bar{W}) = 0$ unless $Z \in II$ and $W \in I$. Hence

$$\begin{aligned}
 (4.34) \quad S_{AB} &= \sum_{c,d \leq p < j} |R(X_{ai}, \overline{U_b}, U_d, \bar{X}_{cj})| + \sum_{c,d,e \leq p < j; d < e} |R(X_{ai}, \overline{U_b}, w_{de}, \bar{X}_{cj})| \\
 &= \sum_{c,d,j} \delta_{ij} \delta_{bd} \delta_{cd} \delta_{ab} + \sum_{c,d,e,j; d < e} \frac{1}{\sqrt{2}} \delta_{ij} (\delta_{bd} \delta_{ab} \delta_{ce} + \delta_{be} \delta_{ab} \delta_{cd}) \\
 &\leq 1 + \frac{1}{\sqrt{2}}(p-1)
 \end{aligned}$$

and we have $S_{AB} < 2n - p + 1$ in this case since $1 < p < n$.

(iv) $A = X_{ai}$, $B = w_{bc}$, $b < c$. Note that $R(X_{ai}, \overline{w_{bc}}, Z, \bar{W}) = 0$ unless $Z \in II$ and $W \in I$. Hence

$$\begin{aligned}
 (4.35) \quad S_{AB} &= \sum_{d,e \leq p < j} |R(X_{ai}, \overline{w_{bc}}, U_e, \bar{X}_{dj})| + \sum_{d,e,e \leq p < j; e < f} |R(X_{ai}, \overline{w_{bc}}, w_{ef}, \bar{X}_{dj})| \\
 &= \sum_{d,e,j} \frac{1}{\sqrt{2}} \delta_{ij} (\delta_{be} \delta_{de} \delta_{ac} + \delta_{ce} \delta_{de} \delta_{ab}) \\
 &\quad + \sum_{d,e,f,j; e < f} \frac{1}{2} \delta_{ij} (\delta_{fc} \delta_{ed} \delta_{ab} + \delta_{be} \delta_{fd} \delta_{ac} + \delta_{ec} \delta_{fd} \delta_{ab} + \delta_{fb} \delta_{ed} \delta_{ca}) \\
 &= \begin{cases} \frac{1}{\sqrt{2}} + \frac{1}{2}(p-1), & \text{if } a = b, i = j; \text{ so } a \neq c \\ \frac{1}{\sqrt{2}} + \frac{1}{2}(p-1), & a = c, i = j; \text{ so } a \neq b \\ 0, & i \neq j; \text{ or } a \neq b \text{ and } a \neq c. \end{cases}
 \end{aligned}$$

and in this case we have $S_{AB} < 2n - p + 1$ since $1 < p < n$.

(v) $A = Y_{ai}$, $B = Y_{bj}$, $(a, i) \neq (b, j)$. This case is similar to case (i)

(vi) $A = Y_{ai}$, $B = U_b$. This case is similar to case (iii).

(vii) $A = Y_{ai}$, $B = w_{bc}$. This case is similar to case (iv).

(viii) $A = U_a$, $B = w_{bc}$, $b < c$. Then

$$\begin{aligned}
 (4.36) \quad S_{AB} &= \sum_{d; e < f} |R(U_a, \bar{w}_{bc}, U_d, \bar{w}_{ef})| + \sum_{(d,e) \neq (f,g); d < e, f < g} |R(U_a, \bar{w}_{bc}, w_{de}, \bar{w}_{fg})| \\
 &\quad + \sum_{(d,i) \neq (e,j)} |R(U_a, \bar{w}_{bc}, X_{di}, \bar{X}_{ej})| + |R(U_a, \bar{w}_{bc}, Y_{di}, \bar{Y}_{ej})| \\
 &= \sum_{d; e < f} (\delta_{ab} \delta_{fd} \delta_{ae} \delta_{cd} + \delta_{ac} \delta_{ed} \delta_{af} \delta_{bd}) \\
 &\quad + \sum_{(d,e) \neq (f,g); d < e; f < g} \left[\frac{1}{\sqrt{2}} \delta_{ab} (\delta_{eg} \delta_{af} \delta_{cd} + \delta_{df} \delta_{ag} \delta_{ce} + \delta_{dg} \delta_{af} \delta_{ce} + \delta_{ef} \delta_{ag} \delta_{cd}) \right. \\
 &\quad \left. + \frac{1}{\sqrt{2}} \delta_{ac} (\delta_{eg} \delta_{af} \delta_{bd} + \delta_{df} \delta_{ag} \delta_{be} + \delta_{dg} \delta_{af} \delta_{be} + \delta_{ef} \delta_{ag} \delta_{bd}) \right] \\
 &\quad + \sum_{(d,i) \neq (e,j)} \left[\frac{1}{\sqrt{2}} \delta_{ij} (\delta_{ab} \delta_{ae} \delta_{cd} + \delta_{ac} \delta_{ae} \delta_{bd}) + \frac{1}{\sqrt{2}} \delta_{ij} (\delta_{ac} \delta_{bd} \delta_{ae} + \delta_{ab} \delta_{cd} \delta_{ae}) \right] \\
 &= I + II + III
 \end{aligned}$$

Note that all terms are zero, if $a \neq b$ and $a \neq c$. Moreover, if $a = b$, then $a \neq c$, and if $a = c$, then $a \neq b$. Hence

$$(4.37) \quad I = \begin{cases} 1, & \text{if } a = b; \\ 1, & \text{if } a = c; \end{cases}$$

$$(4.38) \quad II = \begin{cases} \frac{1}{\sqrt{2}}(p-2), & \text{if } a = b; \\ \frac{1}{\sqrt{2}}(p-2), & \text{if } a = c; \end{cases}$$

$$(4.39) \quad III = \begin{cases} \frac{2}{\sqrt{2}}(n-p), & \text{if } a = b; \\ \frac{2}{\sqrt{2}}(n-p), & \text{if } a = c; \end{cases}$$

Thus

$$(4.40) \quad S_{AB} = \begin{cases} \frac{1}{\sqrt{2}}(2n-p-2) + 1, & \text{if } a = b; \\ \frac{1}{\sqrt{2}}(2n-p-2) + 1, & \text{if } a = c; \\ 0, & \text{if } a \neq b \text{ and } a \neq c, \end{cases}$$

and in this case we have $S_{AB} < 2n - p + 1$ since $1 < p < n$.

(ix) $A = U_a$, $B = U_b$ with $a \neq b$. In this case we have $S_{AB} = 0$.

(x) $A = w_{ab}$, $B = w_{cd}$ with $(a, b) \neq (c, d)$, $a < b$, $c < d$.

(case 1) $a = c$, $b \neq d$. Since $S_{AB} = S_{BA}$, we may assume $b < d$. So $b > a = c < d$. By Lemma 4.5, the only nonzero terms in the sum of S_{AB} are:

$$(4.41) \quad \sum_{e, f \leq p < i, j; (e, i) \neq (f, j)} |R(w_{ab}, \bar{w}_{cd}, X_{ei}, \bar{X}_{fj})| \leq \frac{1}{2}(n-p),$$

$$(4.42) \quad \sum_{e, f \leq p < i, j} |R(w_{ab}, \bar{w}_{cd}, Y_{ei}, \bar{Y}_{fj})| \leq \frac{1}{2}(n-p),$$

$$(4.43) \quad \sum_{e, f, g \leq p; e < f} |R(w_{ab}, \bar{w}_{cd}, U_g, \bar{w}_{ef})| \leq \frac{1}{\sqrt{2}},$$

$$(4.44) \quad \sum_{e, f, g, h \leq p; e < f, g < h} |R(w_{ab}, \bar{w}_{cd}, w_{ef}, \bar{w}_{gh})| \leq \frac{1}{2}(p-2),$$

where (4.44) is obtained from (4.16) and estimating exactly as in (3.36). In fact, the LHS of (4.44) is given by the second line in (3.36) provided the negative signs there are changed to signs there.

So $S_{AB} \leq (n-p) + \frac{1}{2}(p-2) + \frac{1}{\sqrt{2}} < 2n - p + 1$.

(case 2) $a \neq c$, $b = d$. As above, we may assume $a < c$. So $a < b = d > c$. The only nonzero terms in the sum of S_{AB} are:

$$(4.45) \quad \sum_{e, f \leq p < i, j} |R(w_{ab}, \bar{w}_{cd}, X_{ei}, \bar{X}_{fj})| \leq \frac{1}{2}(n-p),$$

$$(4.46) \quad \sum_{e, f \leq p < i, j} |R(w_{ab}, \bar{w}_{cd}, Y_{ei}, \bar{Y}_{fj})| \leq \frac{1}{2}(n-p),$$

$$(4.47) \quad \sum_{e,f,g \leq p; e < f} |R(w_{ab}, \bar{w}_{cd}, \bar{w}_{ef}, U_g)| \leq \frac{1}{\sqrt{2}},$$

$$(4.48) \quad \sum_{e,f,g,h \leq p < e < f, g < h} |R(y_{ab}, \bar{y}_{cd}, y_{ef}, \bar{y}_{gh})| \leq \frac{1}{2}(p-2),$$

where as above, (4.48) is obtained from (4.16) and estimating exactly as in (3.40).

So we also have $S_{AB} < 2n - p + 1$.

(case 3) $a \neq c, b \neq d$. As above, we may assume $b < d$. The only nonzero terms in the sum of S_{AB} are:

$$(4.49) \quad \sum_{e,f \leq p < i,j} |R(w_{ab}, \bar{w}_{cd}, X_{ei}, \bar{X}_{fj})| \leq \frac{1}{2}(n-p),$$

Similarly

$$(4.50) \quad \sum_{e,f \leq p < i,j} |R(w_{ab}, \bar{w}_{cd}, Y_{ei}, \bar{Y}_{fj})| \leq \frac{1}{2}(n-p),$$

$$(4.51) \quad \sum_{e,f,g \leq p; e < f} |R(w_{ab}, \bar{w}_{cd}, U_g, \bar{w}_{ef})| \leq \frac{1}{\sqrt{2}},$$

$$(4.52) \quad \sum_{e,f,g \leq p; e < f} |R(w_{ab}, \bar{w}_{cd}, \bar{w}_{ef}, U_g)| \leq \frac{1}{\sqrt{2}},$$

$$(4.53) \quad \sum_{e,f,g,h \leq p < e < f, g < h} |R(w_{ab}, \bar{w}_{cd}, w_{ef}, \bar{w}_{gh})| \leq \frac{1}{2}(p-2),$$

where as above, (4.53) is obtained from (4.16) and estimating exactly as in (3.44).

So $S_{AB} \leq (n-p) + \frac{1}{2}(p-2) + \frac{2}{\sqrt{2}} < 2n - p + 1$.

We have proved above that $\lambda \leq 2n - p + 1$ provided $5p \leq 4n + 3$. Conversely, suppose $5p > 4n + 3$. Define the vector v by: $x_{ai,aj} = 1 = -y_{ai,aj}$ if $i \neq j$, all other x 's, y 's and all t are zeros. Notice that $x_{ai,aj}$ etc are symmetric and real.

$F(v) = \sum_{A,B,C,D} R_{A\bar{B}C\bar{D}} x_{AB} x_{CD}$ is equal to

$$\begin{aligned}
 (4.54) \quad F(v) &= \sum_{a,c \leq p < i,j,k,l; i \neq j, k \neq l} \left[2R(X_{ai}, \bar{X}_{aj}, X_{ck}, \bar{X}_{cl}) - 2R(X_{ai}, \bar{X}_{aj}, Y_{ck}, \bar{Y}_{cl}) \right] \\
 &= 2p^2(n-p)(n-p-1) - p(n-p)(n-p-1) + p^2(n-p)(n-p-1) \\
 &= (n-p)(n-p-1)(3p^2 - p).
 \end{aligned}$$

On the other hand,

$$\sum_{a \leq p < i,j; i \neq j} (x_{ai,aj}^2 + y_{ai,aj}^2) = 2p(n-p)(n-p-1).$$

Hence $F(v)/|v|^2 = (3p-1)/2$ and we have $F(v) > (2n-p+1)|v|^2$ if $5p > 4n+3$. This completes the proof of (a).

To prove (b), suppose $5p < 4n+3$. Then by the computations from (a), we conclude that $\lambda < 2n-p+1$. To prove (c), note that in the above example we in fact have $F(v) = (2n-p+1)|v|^2$ iff $5p+1 = 4n$. \square

Theorem 4.1. *The Kähler C-space (C_n, α_p) , $n \geq 3$, $1 < p < n$ satisfies $QB \geq 0$ if and only if $5p \leq 4n+3$. Moreover, $QB > 0$ if and only if $5p < 4n+3$.*

Proof. The first statement follows from part (a) of Lemmas 4.6 and 4.7 and Corollary 2.1. For the second statement, note that $QB > 0$ iff $G - F > 0$ on $\Omega_{\mathbb{R}}^{1,1}(p) \setminus \mathbb{R}\omega(p)$ by Lemma 2.3. On the other hand, here $G = (2n-p+1)Id$ on $\Omega_{\mathbb{R}}^{1,1}(p) \setminus \mathbb{R}\omega(p)$, and thus by parts (b) and (c) of Lemmas 4.6 and Lemma 4.7 we have $G - F > 0$ iff $5p < 4n+3$. Thus we have $QB > 0$ if and only if $5p < 4n+3$. \square

5. KÄHLER C-SPACE OF TYPE (D_n, α_p)

In this section we will consider the space (D_n, α_p) , with $n \geq 4$, $1 < p < n-1$. Here $D_n = \mathfrak{so}_{2n+1}\mathbb{C}$, consisting of matrices of the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}, A^t = -D, B^t = -B, C^t = -C.$$

See [9]. This case is similar to the case (B_n, α_p) . Using the notations as in section 3. In fact it is more simple. Again, a Cartan subalgebra \mathfrak{h} is generated by $H_i = E_{i,i} - E_{n+i,n+i}$, $1 \leq i \leq n$ with dual basis L_i . Roots are

$$(5.1) \quad \Delta = \{\pm L_i \pm L_j | 1 \leq i, j \leq n, i \neq j\}.$$

The eigenvectors are

$$\begin{aligned}
 & L_i - L_j : X_{i,j}; i \neq j \\
 & L_i + L_j : Y_{i,j}; i \neq j \\
 & -L_i - L_j : Z_{i,j}; i \neq j
 \end{aligned}
 \tag{5.2}$$

Fundamental roots are:

$$(5.3) \quad \alpha_1 = L_1 - L_2, \alpha_2 = L_2 - L_3, \dots, \alpha_{n-1} = L_{n-1} - L_n, \alpha_n = L_{n-1} + L_n.$$

Positive roots are:

$$(5.4) \quad \Delta^+ = \{L_i + L_j\}_{i < j} \cup \{L_i - L_j\}_{i < j}.$$

Now

$$\begin{aligned}
 (5.5) \quad & L_i + L_j = \alpha_i + \dots + \alpha_{j-1} + 2\alpha_j + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \quad i < j \leq n-2 \\
 & L_i + L_{n-1} = \alpha_i + \dots + \alpha_n, \quad i < n-1 \\
 & L_i + L_n = \alpha_i + \dots + \alpha_{n-2} + \alpha_n, \quad i < n-1 \\
 & L_{n-1} + L_n = \alpha_n \\
 & L_i - L_j = \alpha_i + \dots + \alpha_{j-1}, \quad i < j.
 \end{aligned}$$

Let $1 < p < n-1$,

$$\Delta_p^+(k) = \{\alpha \in \Delta^+ \mid \alpha = k\alpha_p + \sum_{i \neq p} m_i \alpha_i, m_i \geq 0, m_i \in \mathbb{Z}\}.$$

Then

$$(5.6) \quad \Delta_p^+(1) = \{L_i - L_j \mid i \leq p < j\} \cup \{L_i + L_j \mid i \leq p < j\},$$

$$(5.7) \quad \Delta_p^+(2) = \{L_i + L_j \mid i < j \leq p\}.$$

$$\Delta_p^+(3) = \emptyset.$$

$$\begin{aligned}
 (5.8) \quad & \mathfrak{m}_1^+ = \left(\bigoplus_{a \leq p < i} \mathbb{C}X_{a,i} \right) \oplus \left(\bigoplus_{a \leq p < i} \mathbb{C}Y_{a,i} \right) \\
 & \mathfrak{m}_2^+ = \bigoplus_{a < b \leq p} \mathbb{C}Y_{a,b}.
 \end{aligned}$$

We may take the Killing form to be $K(X, Y) = \frac{1}{2} \text{tr}(XY)$.

Lemma 5.1. *Let*

- (I) : $X_{ai}, a \leq p < i; Y_{ai}, a \leq p < i.$ (All vectors in this group are in $\mathfrak{m}_1^+.$)
- (II) : $w_{ab} = \frac{1}{\sqrt{2}}Y_{a,b}, a < b \leq p.$ (All vectors in this group are in $\mathfrak{m}_2^+.$)

The vectors in (I) and (II) form a Weyl frame. Their conjugates are $-X_{ia}, -Z_{ia}, a < p \leq i; -\frac{1}{\sqrt{2}}Y_{a,b}, a < b \leq p$ respectively. The dimension of (D_n, α_p) is $\frac{1}{2}p(4n - 3p - 1).$

Hence the computations in section 3 remains unchanged, except the vectors U_a, V_a do not appear in this case. In this case, $\text{Ric} = (2n - p - 1)g$ and we have:

Theorem 5.1. *The C-space $(D_n, \alpha_p), n \geq 4, 1 < p < n - 1$ satisfies $QB \geq 0$ if and only if $5p + 3 \leq 4n.$ Moreover, $QB > 0$ if and only if $5p + 3 < 4n.$*

Remark 5.1. As in the B and C cases, one can see that (D_n, α_p) does not satisfy $B^\perp \geq 0.$

6. APPENDIX 1: COMPUTATIONS OF CURVATURE OF (B_n, α_p)

Let us compute the curvature, $R(X, \bar{Y}, Z, \bar{W}).$ Recall:

Lemma 6.1. $[E_{ij}, E_{kl}] = \delta_{kj}E_{il} - \delta_{il}E_{kj}$ and $\text{tr}(E_{ij}E_{kl}) = \delta_{il}\delta_{kj}$

1: $R(X, \bar{Y}, Z, \bar{W}) = 0,$ if (i) three of them belong to one group and the remaining vector is in the other group; (ii) X, Z are in one group and Y, W are in the other group; or (iii) $X = Y,$ and $Z \neq W.$

The values of the remaining components of the curvature tensors are given by the following formulas together with their permutations from the properties of $R.$ For $a, b, c, d, e, f, g, h \leq p < i, j, k, l$ we have

2: X, Y, Z, W are in (I): In this case, we have

$$R(X, \bar{Y}, Z, \bar{W}) = -\frac{1}{2}K([X, Z], [\bar{Y}, \bar{W}]) + K([X, \bar{Y}], [Z, \bar{W}])$$

(6.1)

$$\begin{aligned} R(X_{ai}, \bar{X}_{bj}, X_{ck}, \bar{X}_{dl}) &= -\frac{1}{2}K([X_{ai}, X_{ck}], [X_{jb}, X_{ld}]) + K([X_{ai}, X_{jb}], [X_{ck}, X_{ld}]) \\ &= K(\delta_{ij}X_{ab} - \delta_{ab}X_{ji}, \delta_{kl}X_{cd} - \delta_{cd}X_{lk}) \\ &= \delta_{ij}\delta_{kl}\delta_{bc}\delta_{ad} + \delta_{ab}\delta_{cd}\delta_{il}\delta_{kj} \end{aligned}$$

(6.2)

$$\begin{aligned} R(X_{ai}, \bar{X}_{bj}, X_{ck}, \bar{Y}_{dl}) &= -\frac{1}{2}K([X_{ai}, X_{ck}], [X_{jb}, Z_{ld}]) + K([X_{ai}, X_{jb}], [X_{ck}, Z_{ld}]) \\ &= 0. \end{aligned}$$

(6.3)

$$\begin{aligned}
R(X_{ai}, \bar{X}_{bj}, Y_{ck}, \bar{Y}_{dl}) &= -\frac{1}{2}K([X_{ai}, Y_{ck}], [X_{jb}, Z_{ld}]) + K([X_{ai}, X_{jb}], [Y_{ck}, Z_{ld}]) \\
&= -\frac{1}{2}K(-\delta_{ik}Y_{ac}, -\delta_{jl}Z_{bd}) + K(\delta_{ij}X_{ab} - \delta_{ab}X_{ji}, \delta_{kl}X_{cd} + \delta_{cd}X_{kl}) \\
&= -\frac{1}{2}\delta_{ik}\delta_{jl}(\delta_{bc}\delta_{ad} - \delta_{ab}\delta_{cd}) + \delta_{ij}\delta_{kl}\delta_{bc}\delta_{ad} - \delta_{ab}\delta_{cd}\delta_{ik}\delta_{jl} \\
&= -\frac{1}{2}\delta_{ik}\delta_{jl}(\delta_{bc}\delta_{ad} + \delta_{ab}\delta_{cd}) + \delta_{ij}\delta_{kl}\delta_{bc}\delta_{ad}
\end{aligned}$$

(6.4)

$$\begin{aligned}
R(X_{ai}, \bar{Y}_{bj}, X_{ck}, \bar{Y}_{dl}) &= \left(-\frac{1}{2}K([X_{ai}, X_{ck}], [Z_{jb}, Z_{ld}]) + K([X_{ai}, Z_{jb}], [X_{ck}, Z_{ld}]) \right) \\
&= 0.
\end{aligned}$$

(6.5)

$$\begin{aligned}
R(X_{ai}, \bar{Y}_{bj}, Y_{ck}, \bar{Y}_{dl}) &= -\frac{1}{2}K([X_{ai}, Y_{ck}], [Z_{jb}, Z_{ld}]) + K([X_{ai}, Z_{jb}], [Y_{ck}, Z_{ld}]) \\
&= 0.
\end{aligned}$$

(6.6)

$$\begin{aligned}
R(Y_{ai}, \bar{Y}_{bj}, Y_{ck}, \bar{Y}_{dl}) &= -\frac{1}{2}K([Y_{ai}, Y_{ck}], [Z_{jb}, Z_{ld}]) + K([Y_{ai}, Z_{jb}], [Y_{ck}, Z_{ld}]) \\
&= K(\delta_{ij}X_{ab} + \delta_{ab}X_{ij}, \delta_{kl}X_{cd} + \delta_{cd}X_{kl}) \\
&= \delta_{ij}\delta_{kl}\delta_{ad}\delta_{bc} + \delta_{jk}\delta_{il}\delta_{ab}\delta_{cd}.
\end{aligned}$$

(6.7)

$$\begin{aligned}
R(U_a, \bar{U}_b, U_c, \bar{U}_d) &= -\frac{1}{2}K([U_a, U_c], [V_b, V_d]) + K([U_a, V_b], [U_c, V_d]) \\
&= -\frac{1}{2}K(Y_{ac}, Z_{bd}) + K(X_{ab}, X_{cd}) \\
&= -\frac{1}{2}(\delta_{bc}\delta_{ad} - \delta_{ab}\delta_{cd}) + \delta_{ad}\delta_{bc}.
\end{aligned}$$

(6.8)

$$\begin{aligned}
R(X_{ai}, \bar{U}_b, Y_{cj}, \bar{U}_d) &= -\frac{1}{2}K(-\delta_{ij}Y_{ac}, -Z_{bd}) + K([X_{ai}, V_b], [Y_{cj}, V_d]) \\
&= -\frac{1}{2}\delta_{ij}(\delta_{bc}\delta_{ad} + \delta_{ab}\delta_{cd}) \\
&= R(Y_{ai}, \bar{U}_b, X_{cj}, \bar{U}_d).
\end{aligned}$$

(6.9)

$$\begin{aligned}
R(X_{ai}, \bar{X}_{bj}, U_c, \bar{U}_d) &= -\frac{1}{2}K([X_{ai}, U_c], [X_{jb}, \bar{V}_d]) + K([X_{ai}, X_{jb}], [U_c, V_d]) \\
&= \delta_{ij}\delta_{bc}\delta_{ad} \\
&= R(Y_{ai}, \bar{Y}_{bj}, U_c, \bar{U}_d).
\end{aligned}$$

(6.10)

$$R(X_{ai}, \bar{Y}_{bj}, U_c, \bar{U}_d) = 0.$$

(6.11)

$$\begin{aligned}
R(X_{ai}, \bar{U}_b, U_c, \bar{U}_d) &= R(Y_{ai}, \bar{U}_b, U_c, \bar{U}_d) = R(X_{ai}, \bar{Y}_{bj}, U_c, \bar{U}_d) = \\
&= R(X_{ai}, \bar{U}_b, X_{cj}, \bar{U}_d) = R(Y_{ai}, \bar{U}_b, Y_{cj}, \bar{U}_d) = \\
&= R(X_{ai}, \bar{X}_{bj}, X_{ck}, \bar{U}_d) = R(X_{ai}, \bar{X}_{bj}, Y_{ck}, \bar{U}_d) = R(X_{ai}, \bar{Y}_{bj}, X_{ck}, \bar{U}_d) = \\
&= R(X_{ai}, \bar{Y}_{bj}, Y_{ck}, \bar{U}_d) = R(Y_{ai}, \bar{Y}_{bj}, Y_{ck}, \bar{U}_d) = 0
\end{aligned}$$

3: X, Y, Z, W are in **(II)**: In this case, we have

$$R(X, \bar{Y}, Z, \bar{W}) = -K([X, Z], [\bar{Y}, \bar{W}]) + 2K([X, \bar{Y}], [Z, \bar{W}])$$

(6.12)

$$\begin{aligned}
R(Y_{ab}, \bar{Y}_{cd}, Y_{ef}, \bar{Y}_{gh}) &= -K([Y_{ab}, Y_{ef}], [Z_{dc}, Z_{hg}]) + 2K([Y_{ab}, Z_{dc}], [Y_{ef}, Z_{hg}]) \\
&= 2K\left(\delta_{bd}X_{ac} + \delta_{ac}X_{bd} - \delta_{ad}X_{bc} - \delta_{bc}X_{ad}, \right. \\
&\quad \left. \delta_{fh}X_{eg} + \delta_{eg}X_{fh} - \delta_{eh}X_{fg} - \delta_{fg}X_{eh}\right) \\
&= 2\delta_{bd}(\delta_{fh}\delta_{ce}\delta_{ag} + \delta_{eg}\delta_{cf}\delta_{ah} - \delta_{eh}\delta_{cf}\delta_{ag} - \delta_{fg}\delta_{ce}\delta_{ah}) \\
&\quad + 2\delta_{ac}(\delta_{fh}\delta_{de}\delta_{bg} + \delta_{eg}\delta_{df}\delta_{bh} - \delta_{eh}\delta_{df}\delta_{bg} - \delta_{fg}\delta_{de}\delta_{bh}) \\
&\quad + 2\delta_{ad}(-\delta_{fh}\delta_{be}\delta_{cg} - \delta_{eg}\delta_{bf}\delta_{ch} + \delta_{eh}\delta_{bf}\delta_{cg} + \delta_{fg}\delta_{be}\delta_{ch}) \\
&\quad + 2\delta_{bc}(-\delta_{fh}\delta_{de}\delta_{ag} - \delta_{eg}\delta_{df}\delta_{ah} + \delta_{eh}\delta_{df}\delta_{ag} + \delta_{fg}\delta_{de}\delta_{ah}).
\end{aligned}$$

4: X, Y, Z, W are not all in one group. By **1**, **2**, the only nonzero case is X, Y in one group and Z, W in another group. So we may assume X, Y in group (II) and Z, W in group (I). So

$$R(X, \bar{Y}, Z, \bar{W}) = -\frac{1}{3}K([X, Z], [\bar{Y}, \bar{W}]) + K([X, \bar{Y}], [Z, \bar{W}]) = K([X, \bar{Y}], [Z, \bar{W}]),$$

because $[X, Z] \in \mathfrak{m}_3^+$ and is 0.

(6.13)

$$\begin{aligned}
R(w_{ab}, \bar{w}_{cd}, X_{ei}, X_{fj}) &= \frac{1}{2}K([Y_{ab}, Z_{dc}], [X_{ei}, X_{fj}]) \\
&= \frac{1}{2}K(\delta_{bd}X_{ac} + \delta_{ac}X_{bd} - \delta_{ad}X_{bc} - \delta_{bc}X_{ad}, \delta_{ij}X_{ef} - \delta_{ef}X_{ji}) \\
&= \frac{1}{2}\delta_{ij}(\delta_{bd}\delta_{ce}\delta_{af} + \delta_{ac}\delta_{de}\delta_{bf} - \delta_{ad}\delta_{ce}\delta_{bf} - \delta_{bc}\delta_{de}\delta_{af})
\end{aligned}$$

(6.14)

$$\begin{aligned}
R(w_{ab}, \bar{w}_{cd}, X_{ei}, \bar{Y}_{fj}) &= \frac{1}{2}K([Y_{ab}, Z_{dc}], [X_{ei}, Z_{jf}]) \\
&= 0.
\end{aligned}$$

(6.15)

$$\begin{aligned}
R(w_{ab}, \bar{w}_{cd}, Y_{ei}, \bar{Y}_{fj}) &= \frac{1}{2} K([Y_{ab}, Z_{dc}], [Y_{ei}, Z_{jf}]) \\
&= \frac{1}{2} K(\delta_{bd}X_{ac} + \delta_{ac}X_{bd} - \delta_{ad}X_{bc} - \delta_{bc}X_{ad}, \delta_{ij}X_{ef} + \delta_{ef}X_{ij}) \\
&= \frac{1}{2} \delta_{ij} (\delta_{bd}\delta_{ce}\delta_{af} + \delta_{ac}\delta_{de}\delta_{bf} - \delta_{ad}\delta_{ce}\delta_{bf} - \delta_{bc}\delta_{de}\delta_{af})
\end{aligned}$$

(6.16)

$$\begin{aligned}
R(w_{ab}, \bar{w}_{cd}, U_e, \bar{U}_f) &= \frac{1}{2} K([Y_{ab}, Z_{dc}], [U_e, V_f]) \\
&= \frac{1}{2} K(\delta_{bd}X_{ac} + \delta_{ac}X_{bd} - \delta_{ad}X_{bc} - \delta_{bc}X_{ad}, X_{ef}) \\
&= \frac{1}{2} (\delta_{bd}\delta_{ce}\delta_{af} + \delta_{ac}\delta_{de}\delta_{bf} - \delta_{ad}\delta_{ce}\delta_{bf} - \delta_{bc}\delta_{de}\delta_{af}).
\end{aligned}$$

(6.17)

$$R(w_{ab}, \bar{w}_{cd}, X_{ei}, \bar{U}_f) = R(w_{ab}, \bar{w}_{cd}, Y_{ei}, \bar{U}_f) = 0.$$

Now let us compute the eigenvalues of the Ricci tensor:

(6.18)

$$\begin{aligned}
\text{Ric}(X_{ai}, \bar{X}_{ai}) &= \sum_{c \leq p < k} (\delta_{ac} + \delta_{ik}) - \frac{1}{2} \sum_{c \leq p < k} \delta_{ik}(\delta_{ac} + 1) + \sum_{c \leq p < k} \delta_{ac} + \sum_{c \leq p} \delta_{ac} \\
&\quad + \frac{1}{2} \sum_{c < d \leq p} (\delta_{ca} + \delta_{da}) \\
&= n - \frac{1}{2}(1 + p) + (n - p) + 1 + \frac{1}{2}(p - 1) \\
&= 2n - p.
\end{aligned}$$

(6.19)

$$\text{Ric}(Y_{ai}, \bar{Y}_{ai}) = 2n - p$$

(6.20)

$$\begin{aligned}
\text{Ric}(U_a, \bar{U}_a) &= \sum_{b \leq p} [-\frac{1}{2}(\delta_{ab} - 1) + \delta_{ab}] + 2 \sum_{b \leq p < i} \delta_{ab} + \frac{1}{2} \sum_{c < d \leq p < i} (\delta_{ac} + \delta_{ad}) \\
&= \frac{1}{2}(p + 1) + 2(n - p) + \frac{1}{2}(p - 1) \\
&= 2n - p.
\end{aligned}$$

$$\begin{aligned}
(6.21) \quad \text{Ric}(w_{ab}, \bar{w}_{ab}) &= \frac{1}{2} \sum_{e < f \leq p} (\delta_{ae} + \delta_{af} + \delta_{be} + \delta_{bf}) + \frac{1}{2} \sum_{e \leq p < i} (\delta_{ae} + \delta_{be}) \\
&\quad + \frac{1}{2} \sum_{e \leq p < i} (\delta_{ae} + \delta_{be}) + \frac{1}{2} \sum_{e \leq p} (\delta_{ae} + \delta_{be}) \\
&= p - 1 + 2(n - p) + 1 \\
&= 2n - p.
\end{aligned}$$

Hence $\text{Ric} = (2n - p)g$.

7. APPENDIX 2: COMPUTATIONS OF CURVATURE OF (C_n, α_p)

1: $R(X, \bar{Y}, Z, \bar{W}) = 0$, if (i) three of them belong to one group and the remaining vector is in the other group: (ii) if X, Z are in one group and Y, W are in the other group.

The values of the remaining components of the curvature tensors are given by the following formulas together with their permutations from the properties of R . For $a, b, c, d, e, f, g, h \leq p < i, j, k, l$ we have

2: X, Y, Z, W are in **(I)**: In this case, we have

$$R(X, \bar{Y}, Z, \bar{W}) = -\frac{1}{2}K([X, Z], [\bar{Y}, \bar{W}]) + K([X, \bar{Y}], [Z, \bar{W}]).$$

(7.1)

$$\begin{aligned}
R(X_{ai}, \bar{X}_{bj}, X_{ck}, \bar{X}_{dl}) &= -\frac{1}{2}K([X_{ai}, X_{ck}], [X_{jb}, X_{ld}]) + K([X_{ai}, X_{jb}], [X_{ck}, X_{ld}]) \\
&= \delta_{ij}\delta_{kl}\delta_{bc}\delta_{ad} + \delta_{ab}\delta_{cd}\delta_{jk}\delta_{il}.
\end{aligned}$$

(7.2)

$$\begin{aligned}
R(X_{ai}, \bar{X}_{bj}, X_{ck}, \bar{Y}_{dl}) &= -\frac{1}{2}K([X_{ai}, X_{ck}], [X_{jb}, Z_{ld}]) + K([X_{ai}, X_{jb}], [X_{ck}, Z_{ld}]) \\
&= 0.
\end{aligned}$$

(7.3)

$$\begin{aligned}
R(X_{ai}, \bar{X}_{bj}, Y_{ck}, \bar{Y}_{dl}) &= -\frac{1}{2}K([X_{ai}, Y_{ck}], [X_{jb}, Z_{ld}]) + K([X_{ai}, X_{jb}], [Y_{ck}, Z_{ld}]) \\
&= -\frac{1}{2}K(\delta_{ik}Y_{ac}, -\delta_{jl}Z_{bd}) + K(\delta_{ij}X_{ab} - \delta_{ab}X_{ji}, \delta_{kl}X_{cd} + \delta_{cd}X_{kl}) \\
&= \frac{1}{2}\delta_{ik}\delta_{jl}(\delta_{bc}\delta_{ad} + \delta_{ab}\delta_{cd}) + \delta_{ij}\delta_{kl}\delta_{bc}\delta_{ad} - \delta_{ab}\delta_{cd}\delta_{ik}\delta_{jl} \\
&= \frac{1}{2}\delta_{ik}\delta_{jl}(\delta_{bc}\delta_{ad} - \delta_{ab}\delta_{cd}) + \delta_{ij}\delta_{kl}\delta_{bc}\delta_{ad}
\end{aligned}$$

(7.4)

$$R(X_{ai}, \bar{Y}_{bj}, X_{ck}, \bar{Y}_{dl}) = \left(-\frac{1}{2}K([X_{ai}, X_{ck}], [Z_{jb}, Z_{ld}]) + K([X_{ai}, Z_{jb}], [X_{ck}, Z_{ld}]) \right) \\ = 0.$$

(7.5)

$$R(X_{ai}, \bar{Y}_{bj}, Y_{ck}, \bar{Y}_{dl}) = -\frac{1}{2}K([X_{ai}, Y_{ck}], [Z_{jb}, Z_{ld}]) + K([X_{ai}, Z_{jb}], [Y_{ck}, Z_{ld}]) \\ = 0.$$

$$R(Y_{ai}, \bar{Y}_{bj}, Y_{ck}, \bar{Y}_{dl}) = -K([Y_{ai}, Y_{ck}], [Z_{jb}, Z_{ld}]) + K([Y_{ai}, Z_{jb}], [Y_{ck}, Z_{ld}]) \\ = K(\delta_{ij}X_{ab} + \delta_{ab}X_{ij}, \delta_{kl}X_{cd} + \delta_{cd}X_{kl}) \\ = \delta_{ij}\delta_{kl}\delta_{ad}\delta_{bc} + \delta_{jk}\delta_{il}\delta_{ab}\delta_{cd}.$$

(7.6)

3: X, Y, Z, W are in **(II)**: In this case, we have

$$R(X, \bar{Y}, Z, \bar{W}) = -K([X, Z], [\bar{Y}, \bar{W}]) + 2K([X, \bar{Y}], [Z, \bar{W}]).$$

$$R(U_a, \bar{U}_b, U_c, \bar{U}_d) = -K([U_a, U_c], [V_b, V_d]) + 2K([U_a, V_b], [U_c, V_d]) \\ = 2\delta_{ab}\delta_{cd}K(X_{ab}, X_{cd}) \\ = 2\delta_{ab}\delta_{cd}\delta_{bc}\delta_{ad}.$$

(7.7)

(7.8)

$$R(U_a, \bar{U}_b, U_c, w_{de}) = -\frac{1}{\sqrt{2}}K([U_a, U_c], [V_b, Z_{ed}]) + \sqrt{2}K([U_a, V_b], [U_c, Z_{ed}]) \\ = \sqrt{2}(\delta_{ab}\delta_{cd}\delta_{bc}\delta_{ae} + \delta_{ab}\delta_{ce}\delta_{bc}\delta_{ad}) \\ = 0. \text{ (since } d < e \text{)}$$

$$R(U_a, \bar{U}_b, w_{cd}, \bar{w}_{ef}) = -\frac{1}{2}K([U_a, Y_{cd}], [V_b, Y_{fe}]) + K([U_a, V_b], [Y_{cd}, Z_{fe}]) \\ = K(\delta_{ab}X_{ab}, \delta_{df}X_{ce} + \delta_{ce}X_{df} + \delta_{cf}X_{de} + \delta_{de}X_{cf}) \\ = \delta_{ab}(\delta_{df}\delta_{ae}\delta_{bc} + \delta_{ce}\delta_{af}\delta_{bd} + \delta_{cf}\delta_{ae}\delta_{bd} + \delta_{de}\delta_{af}\delta_{bc}) \\ = \delta_{ab}(\delta_{df}\delta_{ae}\delta_{bc} + \delta_{ce}\delta_{af}\delta_{bd})$$

(7.9)

(7.10)

$$R(U_a, \bar{w}_{cd}, U_b, \bar{w}_{ef}) = -\frac{1}{2}K([U_a, U_b], [Z_{dc}, Z_{fe}]) + K([U_a, Z_{dc}], [U_b, Z_{fe}]) \\ = K(\delta_{ca}X_{ad} + \delta_{da}X_{ac}, \delta_{eb}X_{bf} + \delta_{fb}X_{be}) \\ = (\delta_{ac}\delta_{eb}\delta_{af}\delta_{db} + \delta_{ac}\delta_{fb}\delta_{ae}\delta_{db} + \delta_{ad}\delta_{eb}\delta_{af}\delta_{cb} + \delta_{ad}\delta_{fb}\delta_{ae}\delta_{cb}) \\ = (\delta_{ac}\delta_{fb}\delta_{ae}\delta_{db} + \delta_{ad}\delta_{eb}\delta_{af}\delta_{cb})$$

$$\begin{aligned}
 (7.11) \quad R(U_a, \bar{w}_{bc}, w_{de}, \bar{w}_{fg}) &= -\frac{1}{2\sqrt{2}}K([U_a, Y_{de}], [Z_{cb}, Z_{gf}]) + \frac{1}{\sqrt{2}}K([U_a, Z_{cb}], [Y_{de}, Z_{gf}]) \\
 &= \frac{1}{\sqrt{2}}K(\delta_{ba}X_{ac} + \delta_{ca}X_{ab}, \delta_{eg}X_{df} + \delta_{df}X_{eg} + \delta_{dg}X_{ef} + \delta_{ef}X_{dg}) \\
 &= \frac{1}{\sqrt{2}}\delta_{ab}(\delta_{eg}\delta_{af}\delta_{cd} + \delta_{df}\delta_{ag}\delta_{ce} + \delta_{dg}\delta_{af}\delta_{ce} + \delta_{ef}\delta_{ag}\delta_{cd}) \\
 &\quad + \frac{1}{\sqrt{2}}\delta_{ac}(\delta_{eg}\delta_{af}\delta_{bd} + \delta_{df}\delta_{ag}\delta_{be} + \delta_{dg}\delta_{af}\delta_{be} + \delta_{ef}\delta_{ag}\delta_{bd})
 \end{aligned}$$

$$\begin{aligned}
 (7.12) \quad R(w_{ab}, \bar{w}_{cd}, w_{ef}, \bar{w}_{gh}) &= -\frac{1}{4}K([Y_{ab}, Y_{ef}], [Z_{dc}, Z_{hg}]) + \frac{1}{2}K([Y_{ab}, Z_{dc}], [Y_{ef}, Z_{hg}]) \\
 &= \frac{1}{2}K(\delta_{bd}X_{ac} + \delta_{ac}X_{bd} + \delta_{ad}X_{bc} + \delta_{bc}X_{ad}, \\
 &\quad \delta_{fh}X_{eg} + \delta_{eg}X_{fh} + \delta_{eh}X_{fg} + \delta_{fg}X_{eh}) \\
 &= \frac{1}{2}[\delta_{bd}(\delta_{fh}\delta_{ag}\delta_{ce} + \delta_{eg}\delta_{ah}\delta_{cf} + \delta_{eh}\delta_{ag}\delta_{cf} + \delta_{fg}\delta_{ah}\delta_{ce}) \\
 &\quad + \delta_{ac}(\delta_{fh}\delta_{bg}\delta_{de} + \delta_{eg}\delta_{bh}\delta_{df} + \delta_{eh}\delta_{bg}\delta_{df} + \delta_{fg}\delta_{bh}\delta_{de}) \\
 &\quad + \delta_{ad}(\delta_{fh}\delta_{bg}\delta_{ce} + \delta_{eg}\delta_{bh}\delta_{cf} + \delta_{eh}\delta_{bg}\delta_{cf} + \delta_{fg}\delta_{bh}\delta_{ce}) \\
 &\quad + \delta_{bc}(\delta_{fh}\delta_{ag}\delta_{de} + \delta_{eg}\delta_{ah}\delta_{df} + \delta_{eh}\delta_{ag}\delta_{df} + \delta_{fg}\delta_{ah}\delta_{de})]
 \end{aligned}$$

4: X, Y, Z, W are not all in one group. By **1**, **2**, the only nonzero case is X, Y in one group and Z, W in another group. So we may assume X, Y in group (II) and Z, W in group (I). So

$$R(X, \bar{Y}, Z, \bar{W}) = -\frac{1}{3}K([X, Z], [\bar{Y}, \bar{W}]) + K([X, \bar{Y}], [Z, \bar{W}]) = K([X, \bar{Y}], [Z, \bar{W}]),$$

because $[X, Z] = 0$.

$$\begin{aligned}
 (7.13) \quad R(U_a, \bar{U}_b, X_{ci}, \bar{X}_{dj}) &= K([U_a, V_b], [X_{ci}, X_{jd}]) \\
 &= K(\delta_{ab}X_{ab}, \delta_{ij}X_{cd} - \delta_{cd}X_{ji}) \\
 &= \delta_{ab}\delta_{ij}\delta_{ad}\delta_{bc}
 \end{aligned}$$

$$\begin{aligned}
 (7.14) \quad R(U_a, \bar{U}_b, X_{ci}, \bar{Y}_{dj}) &= K([U_a, V_b], [X_{ci}, Z_{dj}]) \\
 &= 0.
 \end{aligned}$$

because $a, b \leq p < i, j$.

$$\begin{aligned}
 (7.15) \quad R(U_a, \bar{U}_b, Y_{ci}, \bar{Y}_{dj}) &= K([U_a, V_b], [Y_{ci}, Z_{dj}]) \\
 &= K(\delta_{ab}X_{ab}, \delta_{cd}X_{ij} + \delta_{ij}X_{cd}) \\
 &= \delta_{ab}\delta_{ij}\delta_{ad}\delta_{bc}.
 \end{aligned}$$

$$\begin{aligned}
(7.16) \quad R(U_a, \bar{w}_{bc}, X_{di}, \bar{X}_{ej}) &= \frac{1}{\sqrt{2}} K([U_a, Z_{cb}], [X_{di}, X_{je}]) \\
&= \frac{1}{\sqrt{2}} K(\delta_{ab}X_{ac} + \delta_{ac}X_{ab}, \delta_{ij}X_{de} - \delta_{de}X_{ji}) \\
&= \frac{1}{\sqrt{2}} \delta_{ij}(\delta_{ab}\delta_{ae}\delta_{cd} + \delta_{ac}\delta_{ae}\delta_{bd})
\end{aligned}$$

$$\begin{aligned}
(7.17) \quad R(U_a, \bar{w}_{bc}, X_{di}, \bar{Y}_{ej}) &= \frac{1}{\sqrt{2}} K([U_a, Z_{cb}], [X_{di}, Z_{je}]) \\
&= 0.
\end{aligned}$$

because $a, b \leq p < i, j$.

$$\begin{aligned}
(7.18) \quad R(w_{ab}, \bar{U}_c, X_{di}, \bar{Y}_{ej}) &= \frac{1}{\sqrt{2}} K([Y_{ab}, V_c], [X_{di}, Z_{je}]) \\
&= 0.
\end{aligned}$$

because $a, b \leq p < i, j$.

$$\begin{aligned}
(7.19) \quad R(U_a, \bar{w}_{bc}, Y_{di}, \bar{Y}_{ej}) &= \frac{1}{\sqrt{2}} K([U_a, Z_{cb}], [Y_{di}, Z_{je}]) \\
&= \frac{1}{\sqrt{2}} K(\delta_{ac}X_{ab} + \delta_{ab}X_{ac}, \delta_{ij}X_{de} + \delta_{de}X_{ij} + \delta_{dj}X_{ie} + \delta_{ie}X_{dj}) \\
&= \frac{1}{\sqrt{2}} \delta_{ij}(\delta_{ac}\delta_{bd}\delta_{ae} + \delta_{ab}\delta_{cd}\delta_{ae})
\end{aligned}$$

$$\begin{aligned}
(7.20) \quad R(w_{ab}, \bar{w}_{cd}, X_{ei}, \bar{X}_{fj}) &= \frac{1}{2} K([Y_{ab}, Z_{dc}], [X_{ei}, X_{jf}]) \\
&= \frac{1}{2} K(\delta_{bd}X_{ac} + \delta_{ac}X_{bd} + \delta_{ad}X_{bc} + \delta_{bc}X_{ad}, \delta_{ij}X_{ef} - \delta_{ef}X_{ij}) \\
&= \frac{1}{2} \delta_{ij}(\delta_{bd}\delta_{af}\delta_{ce} + \delta_{ac}\delta_{bf}\delta_{de} + \delta_{ad}\delta_{bf}\delta_{ce} + \delta_{bc}\delta_{af}\delta_{de})
\end{aligned}$$

$$\begin{aligned}
(7.21) \quad R(w_{ab}, \bar{w}_{cd}, X_{ei}, \bar{Y}_{fj}) &= \frac{1}{2} K([Y_{ab}, Z_{dc}], [X_{ei}, Z_{jf}]) \\
&= 0.
\end{aligned}$$

because $a, b \leq p < i, j$.

$$\begin{aligned}
(7.22) \quad R(w_{ab}, \bar{w}_{cd}, Y_{ei}, \bar{Y}_{fj}) &= \frac{1}{2} K([Y_{ab}, Z_{dc}], [Y_{ei}, Z_{jf}]) \\
&= \frac{1}{2} K(\delta_{db}X_{ac} + \delta_{ac}X_{db} + \delta_{ad}X_{bc} + \delta_{bc}X_{ad}, \\
&\quad \delta_{ef}X_{ij} + \delta_{ij}X_{ef} + \delta_{ej}X_{if} + \delta_{if}X_{ej}) \\
&= \frac{1}{2} \delta_{ij}(\delta_{db}\delta_{af}\delta_{ce} + \delta_{ac}\delta_{df}\delta_{be} + \delta_{ad}\delta_{bf}\delta_{ce} + \delta_{bc}\delta_{af}\delta_{de})
\end{aligned}$$

Now let us compute the eigenvalues of the Ricci tensor:

(7.23)

$$\begin{aligned} \text{Ric}(X_{ai}, \overline{X_{ai}}) &= \sum_{b,j} (\delta_{ab} + \delta_{ik} + \frac{1}{2} \delta_{ik} (\delta_{ab} - 1) + \delta_{ab}) + \sum_b \delta_{ab} + \sum_{b,c} \frac{1}{2} (\delta_{ab} + \delta_{ac}) \\ &= (n - p) + p + \frac{1}{2} (1 - p) + (n - p) + 1 + \frac{1}{2} (p - 1) \\ &= 2n - p + 1. \end{aligned}$$

(7.24)

$$\text{Ric}(Y_{ai}, \overline{Y_{ai}}) = 2n - p + 1.$$

(7.25)

$$\begin{aligned} \text{Ric}(U_a, \overline{U_a}) &= \sum_{b,j} 2\delta_{ab} + \sum_b 2\delta_{ab} + \sum_{b < c} (\delta_{ab} + \delta_{ac}) \\ &= 2(n - p) + 2 + (p - 1) \\ &= 2n - p + 1. \end{aligned}$$

(7.26)

$$\begin{aligned} \text{Ric}(w_{ab}, \overline{w_{ab}}) &= \sum_{c,j} (\delta_{ac} + \delta_{bc}) + \sum_c (\delta_{ac} + \delta_{bc}) + \sum_{c,d} \frac{1}{2} (\delta_{ad} + \delta_{ac} + \delta_{bd} + \delta_{bc}) \\ &= 2(n - p) + 2 + \frac{1}{2} [(a - 1) + (p - a) + (b - 1) + (p - b)] \\ &= 2n - p + 1. \end{aligned}$$

Hence $\text{Ric} = (2n - p + 1)g$.

REFERENCES

- [1] Bando, S., *On the classification of three-dimensional compact Kähler manifolds of nonnegative bisectional curvature*, J. Differential Geom. **19** (1984), no. 2, 283-297, MR0755227, Zbl 0547.53034.
- [2] Besse, A. L., *Einstein manifolds*, Springer-verlag, (1987).
- [3] Bishop, R.L., Goldberg, S. I., *On the second cohomology group of a Kähler manifold of positive curvature*, Proc. Amer. Math. Soc. **16**, (1965), 119-122, MR0172221, Zbl 0125.39403.
- [4] Borel, A., *Kählerian coset spaces of semi-simple Lie groups*, Proc. Nat. Acad. Sci. U.S.A., **40** (1954), 1147-1151.
- [5] Borel, A., *On the curvature tensor of Hermitian symmetric manifolds*, Ann. of Math., **71** (1960), 508-521.
- [6] Borel, A. and Hirzebruch, F., *Characteristic classes and homogeneous spaces I*, Amer. J. Math., **80** (1958), 458-538.
- [7] Chau, A., Tam, L.F., *On quadratic orthogonal bisectional curvature*, J. Diff. Geom., to appear.
- [8] Chen, X. X., *On Kähler manifolds with positive orthogonal bisectional curvature*, Adv. Math. **215** (2007), no. 2, 427-445, MR2355611, Zbl 1131.53038.
- [9] Fulton, W., Harris, J., *Representation theory: a first course*, Springer-Verlag (1991).

- [10] Goldberg, S.I., Kobayashi, S., *Holomorphic bisectional curvature*, J. Differential Geom., **1**, 1967, 225-233, MR0227901, Zbl 0169.53202.
- [11] Gu, H.L., Zhang, Z.H., *An Extension of Mok's Theorem on the Generalized Frankel Conjecture*, Sci. China Math. **53** (2010), no. 5, 1253-1264, MR2653275, Zbl 1204.53058.
- [12] Howard, A., Smyth, B. and Wu, H. *On compact Kähler manifolds of nonnegative bisectional curvature, I*, Acta Math. **147** (1981), 51-56, MR0631087, Zbl 0473.53055.
- [13] Itoh, M., *On curvature properties of Kähler C-spaces*, J. Math. Soc. Japan **30** (1978), no. 1, 3971.
- [14] Li, Q., Wu, D. and Zheng, F., *An example of a compact Kähler manifold with nonnegative quadratic bisectional curvature*, arXiv:1110.1749.
- [15] Mok, N., *The uniformization theorem for compact Kähler manifolds of non-negative bisectional curvature*, J. Differential Geom. **27** (1988), no. 2, 179-214, MR0925119, Zbl 0642.53071.
- [16] Mori, S. *Projective manifolds with ample tangent bundles*, Ann. of Math. (2) **110** (1979), 593-606, MR0554387, Zbl 0423.14006.
- [17] Siu, Y. T., Yau, S. T., *Complex Kähler manifolds of positive bisectional curvature*, Invent. Math. **59** (1980), 189-204.
- [18] Wang, H. C., *Closed manifolds with homogeneous complex structures*, Amer. J. Math., **76** (1954), 1-32.
- [19] Wilking, B., *A Lie algebraic approach to Ricci flow invariant curvature conditions and Harnack inequalities*, to appear in J. Reine Angew. Mathematik.
- [20] Wu, H., *On compact Kähler manifolds of nonnegative bisectional curvature II*, Acta Math. **147** (1981), 57-70, MR0631088, Zbl 0473.53056.
- [21] Wu, D., Yau, S.T., Zheng, F. *A degenerate Monge Ampère equation and the boundary classes of Kähler cones*, Math. Res. Lett. **16** (2009), no.2, 365-374, MR2496750, Zbl 1183.32018.
- [22] Yau, S.-T. (ed), *Seminar on differential geometry*, Princeton University Press; Tokyo: University of Tokyo Press, **1982**, MR0645728, Zbl 0471.00020.
- [23] Zheng, F., *Private communication*.

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